RECENT DEVELOPMENTS IN

MATHEMATICS

Editors:

Prof. Dr. Fatih Yılmaz

Prof. Dr. Metin Orbay

Assoc. Prof. Dr. Vildan Öztürk

Umut Selvi

Samet Arpacı

PEGEM AKADEMİ

PEGEM AKADEMİ

Editors: Fatih YILMAZ - Metin ORBAY - Vildan ÖZTÜRK - Umut SELVİ - Samet ARPACI

RECENT DEVELOPMENTS IN MATHEMATICS

ISBN 978-625-6140-48-6

Pegem Akademi Yayıncılık, 1998 yılından bugüne uluslararası düzeyde düzenli faaliyet yürüten **uluslararası akademik bir yayınevi**dir. Yayımladığı kitaplar; Yükseköğretim Kurulunca tanınan yükseköğretim kurumlarının kataloglarında yer almaktadır. Dünyadaki en büyük çevrimiçi kamu erişim kataloğu olan **WorldCat** ve ayrıca Türkiye'de kurulan **Turcademy.com** tarafından yayınları taranmaktadır, indekslenmektedir. Aynı alanda farklı yazarlara ait 2000'in üzerinde yayını bulunmaktadır. Pegem Akademi Yayınları ile ilgili detaylı bilgilere http://pegem.net adresinden ulaşılabilmektedir.

1. Baskı: Kasım 2024, Ankara

Yayın-Proje: Zeynep Güler
Dizgi-Grafik Tasarım: Pegem Akademi
Kapak Tasarımı: Pegem Akademi

Baskı: Sonçağ Yayıncılık Matbaacılık Reklam San Tic. Ltd. Şti.
İstanbul Cad. İstanbul Çarşısı 48/48 İskitler/Ankara

Yayıncı Sertifika No: 51818
Matbaa Sertifika No: 47865

İletişim

Pegem Akademi: Shira Ticaret Merkezi
Macun Mahallesi 204 Cad. No: 141/33, Yenimahalle/Ankara
Yayınevi: 0312 430 67 50
Dağıtım: 0312 434 54 24
Hazırlık Kursları: 0312 419 05 60
İnternet: www.pegem.net
E-ileti: pegem@pegem.net
WhatsApp Hattı: 0538 594 92 40

Preface

This book has been carefully designed to provide an introduction to developments in Mathematics, recently, for under-graduate and postgraduate students. It includes research from different areas. Each topic is divided into sections of approximately the same length for instructors can easily pace their lectures. All definitions and theorems are stated carefully so that the students will appreciate the precision of language. Moreover the proofs are motivated and developed slowly. We are pleased with the range of topics that we have managed to include.

Many thanks to researchers who contributed directly or indirectly to the completion of this book. I am grateful to referees for their healthy criticism and suggestions to improve the quality and standards.

Prof. Dr. Fatih YILMAZ
Editor in Chief

Editors

Prof. Dr. Fatih Yılmaz
Editor in Chief

He obtained his PhD (2013) in Selçuk University. He is co-author of many research papers, and numerous contributions to workshops and conference talks and organizations. He is Editorial Memberof the Turkish Journal of Mathematics and Computer Science, and participated as Invited Editor in Special Issues of journals. He has also participated in numerous European, National and Regional funding projects, and he obtained some mobility grants to stay at the Central European University (Budapest) and Coimbra University (Coimbra). His current research interests include the applications of matrix theory, graph theory, number theory and mathematical education based on projects and competencies. He serves as a guest editor of Axioms, Entropy and Mathematics. Currently, he is working in Ankara Hacı Bayram Veli University.

Prof. Dr. Metin Orbay

He obtained his PhD (2000) in Ondokuz Mayıs University. He is co-author of many research papers indexed in SCI-Expanded/SSCI, and numerous contributions to national and international books, workshops, conference talks and organizations. He worked as a project manager, researcher and trainer in projects supported by the European Union, TÜBİTAK and BAP units of universities. He received the 2019 Honor Award from the Turkish Physics Association in recognition of his efforts to advance the field of physics in Turkey and his significant role in fostering awareness of science and technology. During his academic career, he served in academic and administrative positions such as head of department, director of the institute, and dean. He was elected and appointed as the president of Amasya University for two terms between 2011-2019. He currently continues his studies in Ankara Hacı Bayram Veli University, Polatlı Faculty of Arts and Sciences, Department of Mathematics.

Assoc. Prof. Dr. Vildan Öztürk

She obtained her PhD (2015) in Gazi University. She is co-author of many indexed research papers, and numerous contributions to national and international workshops, conference talks and organizations. She surved as an editor in Journal of Mathematics and Universal Journal of Mathematics and its Applications. Her current research interets include topology and metric fixed point theory. She currently continues her studies in Ankara Haci Bayram Veli University, Polatli Faculty of Arts and Sciences, Department of Mathematics.

Umut Selvi

He took part in two different international bilateral cooperation projects of Tubitak as a scholar. His research interests include matrix theory, number theory, algebraic geometry, Lie algebras and representation theory. Currently, he is PhD student and working in Ankara Hacı Bayram Veli University.

Samet Arpacı

His research interests include matrix theory, number theory, graph theory, coding theory, lattice theory and fuzzy set theory. He is co-author of some indexed research paper. Currently, he is PhD student and working in Ankara Hacı Bayram Veli University.

Contents

1. ON EUCLIDEAN NORMS OF MIN MATRICES WITH CHEBYSHEV POLYNOMIALS

Umut Selvi **Fatih Yılmaz**

Abstract

In this paper, we consider Min matrices whose elements are Chebyshev polynomials of second, third and fourth kinds. We present some norms of Min matrices whose elements are Chebyshev polynomials. Afterwards, we give some examples.

Keywords. Min matrices, Chebyshev polynomials, Matrix norm.

ORCID: 0000-0001-7873-1979
e-mail: fatih.yilmaz@hbv.edu.tr

ORCID: 0000-0003-1015-0056
e-mail: umut.selvi@hbv.edu.tr

1 Introduction

In [11], the authors defined Min matrices as below:

$$A_{\min} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{bmatrix} \tag{1.1}$$

respectively. This matrix form is characterized by $A_{\min} = [A_{\min\{i,j\}}]_{i,j=1}^{n}$.

Recently, there has been extensive research on Min matrices. For example, in [1], the authors compute eigenvalues for Min matrices. The authors, in [10], give factorization, determinant and inverse of Min matrices. In [7], the authors give some results on r-min matrices.

In [8], the four kinds of Chebyshev polynomials introduced by Pafnuty Chebyshev in 1854, as below:

$$T_n(x) = \cos n\theta \quad \text{when} \;\; x = \cos\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{when} \;\; x = \cos\theta,$$

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \quad \text{when} \;\; x = \cos\theta,$$

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \quad \text{when} \;\; x = \cos\theta.$$

In these equations, the following recurrence relations are satisfy:

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \;\; n > 1,$$

$$U_0(x) = 1, \;\; U_1(x) = 2x, \;\; U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \;\; n > 1,$$

$$V_0(x) = 1, \;\; V_1(x) = -1+2x, \;\; V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \;\; n > 1,$$

$$W_0(x) = 1, \;\; W_1(x) = 1+2x, \;\; W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \;\; n > 1.$$

The norm of a matrix is a non-negative real number. In literature, there exist different methods of computing a matrix norm, but they all follow the same definite characteristics. In [5], the well-known Euclidean norm is defined by

$$\|M\|_{\mathbb{E}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |m_{ij}|^2 \right)^{\frac{1}{2}}.$$

In [9], the spectral norm is defined by

$$\|M\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(M^H M)}$$

where M^H is the conjugate transpose of M and λ_i is the eigenvalue of matrix MM^H; here .

In [6], the Euclidean and the spectral norm of a matrix provide the following inequalities:

$$\frac{1}{\sqrt{n}}\|A\|_{\mathbb{E}} \leq \|A\|_2 \leq \|A\|_{\mathbb{E}}, \tag{1.2}$$

and

$$\|A\|_2 \leq \|A\|_{\mathbb{E}} \leq \sqrt{n}\|A\|_2. \tag{1.3}$$

In this paper, we consider n-square Min matrix whose entries are Chebyshev polynomials, given below:

$$U_{\text{min}} = [U_{\min\{i-1,j-1\}}(x)]_{i,j=1}^n, V_{\text{min}} = [V_{\min\{i-1,j-1\}}(x)]_{i,j=1}^n$$

$$W_{\text{min}} = [W_{\min\{i-1,j-1\}}(x)]_{i,j=1}^n$$

associated with the Chebyshev polynomials of second kind, third kind and fourth kind given in (1.4),(1.5) and (1.6) respectively, i.e.,

$$U_{\text{min}} = \begin{pmatrix} U_0(x) & U_0(x) & U_0(x) & \dots & U_0(x) \\ U_0(x) & U_1(x) & U_1(x) & \dots & U_1(x) \\ U_0(x) & U_1(x) & U_2(x) & \dots & U_2(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_0(x) & U_1(x) & U_2(x) & \dots & U_{n-1}(x) \end{pmatrix} \tag{1.4}$$

and

$$V_{\text{min}} = \begin{pmatrix} V_0(x) & V_0(x) & V_0(x) & \dots & V_0(x) \\ V_0(x) & V_1(x) & V_1(x) & \dots & V_1(x) \\ V_0(x) & V_1(x) & V_2(x) & \dots & V_2(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_0(x) & V_1(x) & V_2(x) & \dots & V_{n-1}(x) \end{pmatrix} \tag{1.5}$$

and

$$W_{\text{min}} = \begin{pmatrix} W_0(x) & W_0(x) & W_0(x) & \dots & W_0(x) \\ W_0(x) & W_1(x) & W_1(x) & \dots & W_1(x) \\ W_0(x) & W_1(x) & W_2(x) & \dots & W_2(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_0(x) & W_1(x) & W_2(x) & \dots & W_{n-1}(x) \end{pmatrix}. \tag{1.6}$$

2 Some Results

In this section, we give some results for norms of $U_{\min}$, $V_{\min}$ and $W_{\min}$.

Theorem 2.1. Let $U_{\min} = [U_{\min\{i-1,j-1\}}(x)]_{i,j=1}^{n}$ be an $n \times n$ matrix. Then, for $|x| < 1$ and $n \geq 2$

$$\|U_{\min}\|_E^2 = \frac{1}{4(1-x^2)}\left[2n^2 + 2n - xU_{2n-1}(x) - 2x^2U_{n-1}^2(x)\right]. \quad (2.2)$$

Proof. According to the definition of the Euclidean norm , we have;

$$\begin{aligned}
||U_{\min}||_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|U_{\min\{i-1,j-1\}}(x)|^2 \\
&= \sum_{k=0}^{n-1}[2(n-1) - 2k + 1]U_k^2(x) \\
&= (2n-1)\sum_{k=0}^{n-1}\frac{\sin^2((k+1)\theta)}{\sin^2\theta} - 2\sum_{k=1}^{n-1}k\frac{\sin^2((k+1)\theta)}{\sin^2\theta} \\
&= \frac{1}{2\sin^2\theta}(2n-1)(n - \cos^2(n\theta) - \frac{\cos\theta\sin(n\theta)\cos(n\theta)}{\sin\theta} + 1) \\
&- \frac{1}{2\sin^2\theta}\left[n^2 + \frac{(2\cos^2\theta - 1)\cos^2(n\theta)}{1-\cos^2\theta} - 2n\cos^2(n\theta)\right. \\
&+ \frac{2\cos\theta\sin(n\theta)\cos(n\theta)}{\sin\theta} - \frac{2n\cos\theta\sin(n\theta)\cos(n\theta)}{\sin\theta} \\
&\left.- \frac{2\cos^2\theta - 1}{-1+\cos^2\theta}\right] \\
&= \frac{1}{2\sin^2\theta}\left[n^2 + n - 1 + \cos^2(n\theta) - \frac{\cos\theta\sin(n\theta)\cos(n\theta)}{\sin\theta}\right. \\
&\left.- \frac{(2\cos^2\theta - 1)(\cos^2(n\theta) - 1)}{-1+\cos^2\theta}\right] \\
&= \frac{1}{2(1-\cos^2\theta)}\left[n^2 + n - \frac{\cos\theta\sin(2n\theta)}{2\sin\theta} - \frac{\cos^2\theta\sin^2(n\theta))}{\sin^2\theta}\right] \\
&= \frac{1}{4(1-x^2)}\left[2n^2 + 2n - xU_{2n-1}(x) - 2x^2U_{n-1}^2(x)\right].
\end{aligned}$$

Thus, proof is completed. □

Example 2.3. Let $n = 2$ in (1.4), i.e. ;

$$U_{\min} = \begin{pmatrix} U_0(x) & U_0(x) \\ U_0(x) & U_1(x) \end{pmatrix}.$$

$$\begin{aligned}||U_{\min}||_E^2 &= \sum_{i=1}^{2}\sum_{j=1}^{2}|U_{\min\{i-1,j-1\}}(x)|^2\\ &= \sum_{k=0}^{1}[3-2k]U_k^2(x)\\ &= 3U_0^2(x)+1U_1^2(x)\\ &= 3(1^2)+1(2x)^2\\ &= 3+4x^2\end{aligned}$$

and, for $n=2$ in (2.2), we have;

$$\begin{aligned}||U_{\min}||_E^2 &= \frac{1}{4(1-x^2)}\left[2(2^2)+2(2)-xU_3(x)-2x^2U_1^2(x)\right]\\ &= \frac{1}{4(1-x^2)}\left[12-x(8x^3-4x)-2x^2(8x^3-4x)^2)\right]\\ &= \frac{1}{4(1-x^2)}\left[12+4x^2-16x^4\right]\\ &= \frac{1}{4(1-x^2)}\left[4(3+4x^2)(1-x^2)\right]\\ &= 3+4x^2.\end{aligned}$$

Corollary 2.4. Let $U_{\min}=[U_{\min\{i-1,j-1\}}(x)]_{i,j=1}^{n}$ be the matrix defined in (1.4), then;

$$\frac{1}{2\sqrt{n}}\left[\frac{1}{4(1-x^2)}\left(2n^2+2n-xU_{2n-1}(x)-2x^2U_{n-1}^2(x)\right)\right]^{\frac{1}{2}} \leq \|U_{\min}\|_2 \tag{2.5}$$

and

$$||U_{\min}||_2 \leq \frac{1}{2}\left[\frac{1}{4(1-x^2)}\left(2n^2+2n-xU_{2n-1}(x)-2x^2U_{n-1}^2(x)\right)\right]^{\frac{1}{2}}. \tag{2.6}$$

Proof. By using (1.2) and (2.1) , the proof is clearly obtained. □

Theorem 2.7. Let $V_{\min}=[V_{\min\{i-1,j-1\}}(x)]_{i,j=1}^{n}$ be an $n\times n$ matrix. Then, for $|x|<1$ and $n\geq 2$

$$\|V_{\min}\|_E^2 = \frac{1}{x+1}\left(n^2+xU_{n-1}^2(x)\right). \tag{2.8}$$

Proof. According to the definition of the Euclidean norm , we have;

$$
\begin{aligned}
||V_{\text{min}}||_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|V_{\text{min}\{i-1,j-1\}}(x)|^2 \\
&= \sum_{k=0}^{n-1}[2(n-1)-2k+1]V_k^2(x) \\
&= (2n-1)\sum_{k=0}^{n-1}\frac{\cos^2((k+\frac{1}{2})\theta)}{\cos^2(\frac{\theta}{2})} - 2\sum_{k=1}^{n-1}k\frac{\cos^2((k+\frac{1}{2})\theta)}{\cos^2(\frac{\theta}{2})} \\
&= (2n-1)\left(\frac{1}{2}\frac{n}{\cos^2(\frac{\theta}{2})} + \frac{1}{4}\frac{\sin(k\theta)\cos^2(k\theta))}{\sin(\frac{\theta}{2})\cos^3(\frac{\theta}{2}))}\right) \\
&\quad - \frac{-n+n^2+\frac{\cos(\theta)(1-\cos^2(n\theta))}{-1+\cos^2(\theta)} - \frac{\sin(n\theta)\cos(n\theta)(1-2n)}{\sin(\theta)}}{\cos(\theta)+1} \\
&= \frac{1}{\cos(\theta)+1}\left(2n^2-n+\frac{(2n-1)}{2}U_{2n-1}(x)\right) \\
&\quad - \frac{1}{\cos(\theta)+1}\left(-n+n^2-\cos(\theta)U_{n-1}^2(x)+U_{2n-1}\frac{2n-1}{2}\right).
\end{aligned}
$$

Thus, proof is completed. □

Example 2.9. Let n = 3 in (1.5), i.e.,

$$
V_{\text{min}} = \begin{pmatrix} V_0(x) & V_0(x) & V_0(x) \\ V_0(x) & V_1(x) & V_1(x) \\ V_0(x) & V_1(x) & V_2(x) \end{pmatrix}.
$$

The Euclidean norm of V_{min} is

$$
\begin{aligned}
||V_{\text{min}}||_E^2 &= \sum_{i=1}^{3}\sum_{j=1}^{3}|V_{\text{min}\{i-1,j-1\}}(x)|^2 \\
&= \sum_{k=0}^{2}[5-2k+1]V_k^2(x) \\
&= 5(1)^2+3(-1+2x)^2+1(-1-2x+4x^2)^2 \\
&= 9-8x+8x^2-16x^3+16x^4
\end{aligned}
$$

and, for n=3 in (2.8), we obtain;

$$
\begin{aligned}
||V_{\min}||_E^2 &= \frac{1}{x+1}\left(3^2 + xU_2^2(x)\right) \\
&= \frac{1}{x+1}\left(9 + x(4x^2-1)^2\right) \\
&= \frac{1}{x+1}\left(9 + x - 8x^3 + 16x^5\right).
\end{aligned}
$$

Corollary 2.10. Let $V_{\min} = [V_{\min\{i-1,j-1\}}(x)]_{i,j=1}^n$ be the matrix defined in (1.5), then;

$$\frac{1}{2\sqrt{n}}\left[\frac{1}{x+1}\left(n^2 + xU_{n-1}^2(x)\right)\right]^{\frac{1}{2}} \leq \|V_{\min}\|_2 \tag{2.11}$$

and

$$\|V_{\min}\|_2 \leq \frac{1}{2}\left[\frac{1}{x+1}\left(n^2 + xU_{n-1}^2(x)\right)\right]^{\frac{1}{2}}.$$

Proof. By using (1.2) and (2.7) , the proof is clearly obtained. □

Theorem 2.12. Let $W_{\min} = [W_{\min\{i-1,j-1\}}(x)]_{i,j=1}^n$ be an $n \times n$ matrix. Then, for $|x| < 1$ and $n \geq 2$

$$||W_{\min}||_E^2 = \frac{1}{(1-x)}\left(n^2 - xU_{n-1}^2(x)\right). \tag{2.13}$$

Proof. According to the definition of the Euclidean norm , we have

$$
\begin{aligned}
||W_{\min}||_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|W_{\min\{i-1,j-1\}}(x)|^2 \\
&= \sum_{k=0}^{n-1}[2(n-1)-2k+1]W_k^2(x) \\
&= (2n-1)\sum_{k=0}^{n-1}\frac{\sin^2((k+\frac{1}{2})\theta)}{\sin^2(\frac{\theta}{2})} - 2\sum_{k=1}^{n-1}k\frac{\sin^2((k+\frac{1}{2})\theta)}{\sin^2(\frac{\theta}{2})} \\
&= \frac{(2n-1)}{1-\cos(\theta)}\left(n-\frac{1}{2}\frac{\sin(2n\theta)}{\sin(\theta)}\right) - \frac{1}{1-\cos(\theta)}\Bigg[-(n-1)n \\
&- (-2n+2n^2+\frac{\cos(\theta)(1-\cos(n\theta)^2)}{-1+\cos(\theta)^2} \\
&- \frac{\sin(n\theta)\cos(n\theta)(1-2n)}{\sin(\theta)})\Bigg] \\
&= \frac{1}{1-\cos(\theta)}\left(2n^2-n-\frac{(2n-1)}{2}U_{2n-1}(x)\right) \\
&- \frac{1}{1-\cos(\theta)}\Bigg(-n^2+n-2n+2n^2+\cos(\theta)U_{n-1}^2(x) \\
&- \frac{(2n-1)}{2}U_{2n-1}\Bigg).
\end{aligned}
$$

Thus, proof is completed. □

Example 2.14. Let n = 4 in (1.6), i.e.,

$$
W_{\min} = \begin{pmatrix} W_0(x) & W_0(x) & W_0(x) & W_0(x) \\ W_0(x) & W_1(x) & W_1(x) & W_1(x) \\ W_0(x) & W_1(x) & W_2(x) & W_2(x) \\ W_0(x) & W_1(x) & W_2(x) & W_3(x) \end{pmatrix}.
$$

The Euclidean norm of $W_{\min}$ is

$$\begin{aligned}
||W_{\min}||_E^2 &= \sum_{i=1}^{4}\sum_{j=1}^{4} |W_{\min\{i-1,j-1\}}(x)|^2 \\
&= \sum_{k=0}^{3}[7-2k]W_k^2(x) \\
&= 7W_0^2(x) + 5W_1^2(x) + 3W_2^2(x) + 1W_3^2(x) \\
&= 7(1)^2 + 5(1+2x)^2 + 3(-1+2x+4x^2)^2 \\
&+ 1(-1-4x+4x^2+8x^3)^2 \\
&= 16 + 16x + 16x^2 + 64x^5 + 64x^6
\end{aligned}$$

and for n = 4 in (2.13), we get,

$$\begin{aligned}
||W_{\min}||_E^2 &= \frac{1}{(1-x)}\left(4^2 - xU_3^2(x)\right) \\
&= \frac{1}{(1-x)}\left(16 - x(-4x^2+8x^3)^2\right) \\
&= \frac{1}{(1-x)}\left(16 - 16x^3 + 64x^5 - 64x^7\right).
\end{aligned}$$

Corollary 2.15. Let $W_{\min} = [W_{\min\{i-1,j-1\}}(x)]_{i,j=1}^{n}$ be the matrix defined in (1.6), then;

$$\frac{1}{2\sqrt{n}}\left[\frac{1}{(1-x)}\left(n^2 - xU_{n-1}^2(x)\right)\right]^{\frac{1}{2}} \leq \|W_{\min}\|_2 \tag{2.16}$$

and

$$\|W_{\min}\|_2 \leq \frac{1}{2}\left[\frac{1}{(1-x)}\left(n^2 - xU_{n-1}^2(x)\right)\right]^{\frac{1}{2}}.$$

Proof. By using (1.2) and (2.12) , the proof is clearly obtained. □

3 Conclusion

In this paper, we construct Min matrices with elements defined through Chebyshev polynomials. We hope that these insights will encourage further research on Min matrices and Chebyshev polynomials, with potential applications across various fields of mathematics.

References

[1] Bhatia, R. (2011). *Min Matrices and Mean Matrices*. Mathematical Intelligencer, 33(2).

[2] Faber, V., Liesen, J., & Tichý, P. (2010). *On Chebyshev polynomials of matrices*. SIAM journal on Matrix Analysis and Applications, 31(4), 2205-2221.

[3] da Fonseca, C. M., Kizilates, C., & Terzioglu, N. (2024). *A new generalization of min and max matrices and their reciprocals counterparts*. Filomat, 3, 421-435.

[4] Horn, R. A., & Johnson, C. R. (2012). *Matrix analysis*. Cambridge University Press.

[5] Jafari-Petroudi, S. H., & Pirouz, B. (2015). *A particular matrix, its inversion and some norms*. Applied and Computational Mathematics, 4(2), 47-52.

[6] Jafari-Petroudi, S., & Pirouz, M. (2016). *On the bounds for the spectral norm of particular matrices with Fibonacci and Lucas numbers*. Int. J. Adv. Appl. Math. and Mech, 3(4), 82-90.

[7] Kızılateş, C., & Terzioğlu, N. (2022). *On r-min and r-max matrices*. Journal of Applied Mathematics and Computing, 68(6), 4559-4588.

[8] Mason, J. C., & Handscomb, D. C. (2002). *Chebyshev polynomials*. Chapman and Hall/CRC.

[9] Mathias, R. (1990). *The spectral norm of a nonnegative matrix*. Linear algebra and its applications, 139, 269-284.

[10] Mattila, M., & Haukkanen, P. (2016). Studying the various properties of Min and Max matrices-elementary vs. more advanced methods. Special Matrices, 4(1).

[11] Pólya, G., & Szegö, G. (1972). *Problems and Theorems in Analysis: Series, integral calculus, theory of functions*. (Vol. 1). Berlin: Springer.

[12] Rivlin, T. J. (2020). *Chebyshev polynomials*. Courier Dover Publications.

[13] Yilmaz, F., Arpaci, S., & Ertaş, A. (2024). *Some results for min matrices associated with Chebyshev polynomials*. Logic Journal of the IGPL, jzae028.

2. THE GENERALIZED FIBONACCI-FRANK AND LUCAS-FRANK MATRICES

Efruz Özlem Mersin

Abstract

The Frank matrix is a special maximum matrix that is important in the literature in terms of its eigenvalues. In the present paper, we introduce some new generalizations of the Frank matrix which are called the generalized Fibonacci-Frank and Lucas-Frank matrices. We explore upper bounds for the largest eigenvalues of the newly defined matrices. We also examine the Euclidean and spectral norms of these matrices.

Keywords. Fibonacci number sequence, Frank matrix, Lucas number sequence, Norm

ORCID: 0000-0001-6260-9063
e-mail: efruzmersin@aksaray.edu.tr

1 Introduction

Matrices play a significant role in solving and modelling problems across various disciplines, given their extensive range of applications. Quantities such as the eigenvalues, spectral radius, determinant, trace, and norm of a matrix provide crucial information about the matrix in question. These quantities constitute key areas of study in linear algebra and matrix theory. Undoubtedly, investigating these quantities on special matrices allows for more meaningful results. The Frank matrix, notable for its eigenvalues, is a special type of maximum matrix with significant relevance in the literature. The rth order Frank matrix is defined as [1]

$$F_r = [p_{uv}]_{u,v=1}^{r} = \begin{bmatrix} r & r-1 & 0 & 0 & \dots & 0 & 0 \\ r-1 & r-1 & r-2 & 0 & \dots & 0 & 0 \\ r-2 & r-2 & r-2 & r-3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

The entries of the Frank matrix F_r are described by the rule

$$p_{uv} = \begin{cases} r+1-\max(u,v), & u > v-2 \\ 0, & \text{or else.} \end{cases}$$

There are various studies in the literature on Frank matrix [1, 2, 3, 4, 5, 6]. The eigenvalues of the Frank matrix F_r appear in real, positive, and reciprocal pairs [2]. One of its eigenvalues is 1, when r is odd. Additionally, $\det(F_r) = 1$ for all values of r [2].

On the other hand, number sequences have numerous applications in mathematical modelling, computer science, statistics, physics, and other sciences. The Fibonacci and Lucas number sequences are the most popular number sequences, and they are defined by the recurrence relations $f_s = f_{s-1} + f_{s-2}$ with $f_0 = 0$, $f_1 = 1$ and $l_s = l_{s-1} + l_{s-2}$ with $l_0 = 2$, $l_1 = 1$ [7]. The Binet formulas for the Fibonacci and Lucas number sequences are

$$f_s = \frac{\varphi^s - \psi^s}{\varphi - \psi},$$
$$l_s = \varphi^s + \psi^s,$$

respectively, where φ and ψ are the roots of the characteristic equation $\lambda^2 - \lambda - 1 = 0$ [7]. For the Fibonacci and Lucas number sequences, the following recurrence relations hold true [7, 8]

$$\sum_{t=1}^{s} f_t = f_{s+2} - 1, \quad \sum_{t=1}^{s} f_t^2 = f_s f_{s+1},$$

$$\sum_{t=1}^{s} l_t = l_{s+2} - 3, \quad \sum_{t=1}^{s} l_t^2 = l_s l_{s+1} - 2.$$

There exist numerous studies in the literature concerning the eigenvalues and norms of matrices involving the Fibonacci and Lucas number sequences [9, 10, 11, 12, 13]. By using the real number sequence

$a = (a_1, a_2, a_3, \ldots, a_r)$ instead of $1, 2, 3, \ldots, r$ in the Frank matrix F_r, the generalized Frank matrix $F_{a_r} = [q_{uv}]_{u,v=1}^{r}$ has been defined, whose entries are described by [4]

$$q_{uv} = \begin{cases} a_{r+1-\max(u,v)}, & u > v-2 \\ 0, & \text{or else.} \end{cases}$$

The algebraic structure of the generalized Frank matrix F_{a_r} has been examined, and its properties such as determinant, inverse, characteristic polynomial and LU decomposition have been obtained [4]. The special cases of the generalized Frank matrix have been defined using the Fibonacci sequence $f = (f_2, f_3, f_4, \ldots, f_{r+1})$ and Lucas sequence $l = (l_1, l_2, l_3, \ldots, l_r)$ as the real number sequence $a = (a_1, a_2, a_3, \ldots, a_r)$ in the generalized Frank matrix F_{a_r}. These special cases are known as the Fibonacci-Frank matrix F_{f_r} and Lucas-Frank matrix F_{l_r}, respectively [14]. Properties such as the number of eigenvalues within a specified interval, bounds for the largest eigenvalues, and some norms of these matrices have also been investigated [15].

In this paper, we introduce the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$ and generalized Lucas-Frank matrix $F_{G_{l_r}}$ using the sequences $f^* = (f_s, f_{s+1}, f_{s+2}, \ldots, f_{s+r-1})$ and $l^* = (l_s, l_{s+1}, l_{s+2}, \ldots, l_{s+r-1})$, respectively, as the real number sequence $a = (a_1, a_2, a_3, \ldots, a_r)$ in the generalized Frank matrix F_{a_r}. Here, f_s and l_s represent the sth Fibonacci and Lucas numbers, respectively.

The generalized Fibonacci-Frank matrix $F_{G_{f_r}} = [f_{uv}]_{u,v=1}^{r}$ is defined as

$$F_{G_{f_r}} = \begin{bmatrix} f_{s+r-1} & f_{s+r-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{s+r-2} & f_{s+r-2} & f_{s+r-3} & 0 & \cdots & 0 & 0 \\ f_{s+r-3} & f_{s+r-3} & f_{s+r-3} & f_{s+r-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+1} & f_{s+1} & f_{s+1} & f_{s+1} & \cdots & f_{s+1} & f_s \\ f_s & f_s & f_s & f_s & \cdots & f_s & f_s \end{bmatrix},$$

where f_s is the sth Fibonacci number. The entries of the matrix $F_{G_{f_r}}$ are described by the rule

$$f_{uv} = \begin{cases} f_{s+r-\max(u,v)}, & u > v-2 \\ 0, & \text{or else.} \end{cases}$$

The generalized Lucas-Frank matrix $F_{G_{l_r}} = [l_{uv}]_{u,v=1}^{r}$ is defined as

$$F_{G_{l_r}} = \begin{bmatrix} l_{s+r-1} & l_{s+r-2} & 0 & 0 & \dots & 0 & 0 \\ l_{s+r-2} & l_{s+r-2} & l_{s+r-3} & 0 & \dots & 0 & 0 \\ l_{s+r-3} & l_{s+r-3} & l_{s+r-3} & l_{s+r-4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{s+1} & l_{s+1} & l_{s+1} & l_{s+1} & \dots & l_{s+1} & l_s \\ l_s & l_s & l_s & l_s & \dots & l_s & l_s \end{bmatrix},$$

where l_s is the sth Lucas number. Its entries are described by

$$l_{uv} = \begin{cases} l_{s+r-\max(u,v)}, & u > v-2 \\ 0, & \text{or else.} \end{cases}$$

We note that for $s = 2$ and $s = 1$, the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$ turn into matrices F_{f_r} and F_{l_r}, respectively. By using the findings of [4], we obtain determinants, inverses, LU decompositions and characteristic polynomials of the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$ as follows.

The determinants of the rth order matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$ are calculated by the rules

$$\det\left(F_{G_{f_r}}\right) = f_s \prod_{t=2}^{r} f_{s+t-3}$$

and

$$\det\left(F_{G_{l_r}}\right) = l_s \prod_{t=2}^{r} l_{s+t-3}.$$

The matrix $B_{G_{f_r}} = [\beta_{uv}]_{u,v=1}^{r}$ which is the inverse of the rth order matrix

$F_{G_{f_r}}$ has the following entries

$$\beta_{uv} = \begin{cases} \dfrac{1}{f_{s+r-3}}, & u=v=1 \\ \dfrac{f_{s+1}}{f_s f_{s-1}}, & u=v=r \\ \dfrac{f_{s+r-u+1}}{f_{s+r-u-1}f_{s+r-u-2}}, & u=v, \quad u,v \neq 1,r \\ -\dfrac{1}{f_{s+r-u-1}}, & u=v+1 \\ 0, & u>v+1 \\ (-1)^{v-u}\displaystyle\prod_{m=1}^{v-u}\beta_{uu}\dfrac{f_{s+r-u-m}}{f_{s+r-u-m-2}}, & u<v<r \\ -\beta_{u,r-1}, & u<v=r. \end{cases}$$

The entries of the matrix $H_{G_{l_r}} = [\eta_{uv}]_{u,v=1}^{r}$, which is the inverse of the rth order matrix $F_{G_{l_r}}$, are described by the rule

$$\eta_{uv} = \begin{cases} \dfrac{1}{l_{s+r-3}}, & u=v=1 \\ \dfrac{l_{s+1}}{l_s l_{s-1}}, & u=v=r \\ \dfrac{l_{s+r-u+1}}{l_{s+r-u-1}l_{s+r-u-2}}, & u=v, \quad u,v \neq 1,r \\ -\dfrac{1}{l_{s+r-u-1}}, & u=v+1 \\ 0, & u>v+1 \\ (-1)^{v-u}\displaystyle\prod_{m=1}^{v-u}\eta_{uu}\dfrac{l_{s+r-u-m}}{l_{s+r-u-m-2}}, & u<v<r \\ -\eta_{u,r-1}, & u<v=r. \end{cases}$$

The LU decomposition of the matrix $F_{G_{f_r}}$ exists for all r, and it consists of

$L=[\tau_{uv}]_{u,v=1}^{r}$ and $U=[\vartheta_{uv}]_{u,v=1}^{r}$, such that

$$\tau_{uv}=\begin{cases}0, & u<v\\ 1, & u=v\\ \dfrac{f_{s+r-u}}{f_{s+r-v}}, & \text{or else}\end{cases}$$

and

$$\vartheta_{uv}=\begin{cases}f_{s+r-1}, & u=v=1\\ \dfrac{f_{s+r-u}f_{s+r-u-1}}{f_{s+r-u+1}}, & u=v\neq 1\\ f_{s+r-u-1}, & u=v-1\\ 0, & \text{or else.}\end{cases}$$

The LU decomposition of the matrix $F_{G_{l_r}}$ exists for all r, and its components $L=[\tau^*_{uv}]_{u,v=1}^{r}$ and $U=[\vartheta^*_{uv}]_{u,v=1}^{r}$ are as

$$\tau^*_{uv}=\begin{cases}0, & u<v\\ 1, & u=v\\ \dfrac{l_{s+r-u}}{l_{s+r-v}}, & \text{or else}\end{cases}$$

and

$$\vartheta^*_{uv}=\begin{cases}l_{s+r-1}, & u=v=1\\ \dfrac{l_{s+r-u}l_{s+r-u-1}}{l_{s+r-u+1}}, & u=v\neq 1\\ l_{s+r-u-1}, & u=v-1\\ 0, & \text{or else.}\end{cases}$$

The characteristic polynomial of the matrix $F_{G_{f_r}}$ has the recurrence relation

$$P_r(\lambda)=(\lambda-f_{s+r-3})P_{r-1}(\lambda)-f_{s+r-2}\lambda P_{r-2}(\lambda),$$

with

$$P_1(\lambda)=\lambda-f_s \quad \text{and} \quad P_2(\lambda)=\lambda^2-f_{s+2}\lambda+f_sf_{s+1}-f_s^2.$$

Let the polynomial $P_r(\lambda)=\lambda^r+\gamma_{r-1}^{(r)}\lambda^{r-1}+\ldots+\gamma_1^{(r)}\lambda+\gamma_0^{(r)}$ be the characteristic polynomial of the rth order matrix $F_{G_{f_r}}$. Then, there are the following relations between the coefficients of the characteristic polynomial for $1 \leq t \leq r-2$

$$\gamma_0^{(r)} = -f_{s+r-3}\gamma_0^{(r-1)} = (-1)^r \det\left(F_{G_{f_r}}\right),$$

$$\gamma_{r-1}^{(r)} = \gamma_{r-2}^{(r-1)} - f_{s+r-1} = -tr\left(F_{G_{f_r}}\right),$$

and

$$\gamma_t^{(r)} = \gamma_{t-1}^{(r-1)} + f_{s+r-3}\gamma_t^{(r-1)} - f_{s+r-2}\gamma_{t-1}^{(r-2)}.$$

The matrix $F_{G_{l_r}}$ has the characteristic polynomial with the following recurrence relation

$$P_r(\mu) = (\mu - l_{s+r-3}) P_{r-1}(\mu) - l_{s+r-2}\mu P_{r-2}(\mu),$$

$$P_1(\mu) = \mu - l_s \quad \text{and} \quad P_2(\mu) = \mu^2 - l_{s+2}\mu + l_s l_{s+1} - l_s^2.$$

The relations

$$\zeta_0^{(r)} = -l_{s+r-3}\zeta_0^{(r-1)} = (-1)^r \det\left(F_{G_{l_r}}\right),$$

$$\zeta_{r-1}^{(r)} = \zeta_{r-2}^{(r-1)} - l_{s+r-1} = -tr\left(F_{G_{l_r}}\right),$$

and

$$\zeta_t^{(r)} = \zeta_{t-1}^{(r-1)} + l_{s+r-3}\zeta_t^{(r-1)} - l_{s+r-2}\zeta_{t-1}^{(r-2)}$$

hold true for the characteristic polynomial $P_r(\mu) = \mu^r + \zeta_{r-1}^{(r)}\mu^{r-1} + \ldots + \zeta_1^{(r)}\mu + \zeta_0^{(r)}$ of the rth order matrix $F_{G_{l_r}}$, where $1 \leq t \leq r-2$.

In the next section we investigate upper bounds for the largest eigenvalues of the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$.

2 On the largest eigenvalues of the generalized Fibonacci-Frank and Lucas-Frank matrices

Theorem 2.1. The following equalities hold true for the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$

(1) $trF_{G_{f_r}} = f_{s+r+1} - f_{s+1}$,

(2) $trF_{G_{f_r}}^2 = 3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2$,

(3) $$\sum_{t=1}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)^2 = 3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2 - \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right)^2,$$

where λ_t's $(t = 1, 2, \ldots, r)$ are the eigenvalues of the matrix $F_{G_{f_r}}$.

Proof. (1) For the matrix $F_{G_{f_r}} = [f_{uv}]_{u,v=1}^{r}$, we obtain

$$\begin{aligned} trF_{G_{f_r}} = \sum_{t=s}^{s+r-1} f_t &= \sum_{t=1}^{s+r-1} f_t - \sum_{t=1}^{s-1} f_t \\ &= (f_{s+r+1} - 1) - (f_{s+1} - 1) \\ &= f_{s+r+1} - f_{s+1}. \end{aligned}$$

(2) For the matrix $F_{G_{f_r}}^2 = \left[f_{uv}^{(2)}\right]_{u,v=1}^{r}$, we have

$$trF_{G_{f_r}}^2 = \sum_{t=1}^{r} f_{tt}^{(2)} = \sum_{t=1}^{r}\left(\sum_{h=1}^{r} f_{th}f_{ht}\right),$$

where

$$f_{th} = \begin{cases} f_{r+s-\max(t,h)}, & t > h-2 \\ 0, & \text{or else} \end{cases}$$

and

$$f_{ht} = \begin{cases} f_{r+s-\max(h,t)}, & h > t-2 \\ 0, & \text{or else.} \end{cases}$$

If $|t-h| < 2$, then $f_{th}f_{ht} \neq 0$, otherwise $f_{th}f_{ht} = 0$. $|t-h| < 2$ requires the equations $t = h$, $t = h-1$ and $t = h+1$ for $1 < t < r$. Since $f_{11}^{(2)} = f_{s+r-1}^2 + f_{s+r-2}^2$ and $f_{rr}^{(2)} = f_s^2 + f_s^2$, we obtain

$$\begin{aligned} trF_{G_{f_r}}^2 &= \sum_{t=1}^{r} f_{tt}{}^{(2)} = f_{s+r-1}^2 + f_{s+r-2}^2 + \sum_{t=2}^{r-1}\left(\sum_{h=t-1}^{t+1} f_{th}f_{ht}\right) + f_s^2 + f_s^2 \\ &= \sum_{t=2}^{r-1}\left(2f_{s+r-t}^2 + f_{s+r-1-t}^2\right) + f_{s+r-1}^2 + f_{s+r-2}^2 + 2f_s^2 \\ &= 2\sum_{t=2}^{r-1} f_{s+r-t}^2 + \sum_{t=2}^{r-1} f_{s+r-1-t}^2 + f_{s+r-1}^2 + f_{s+r-2}^2 + 2f_s^2 \\ &= 2\left(f_{s+r-2}^2 + f_{s+r-3}^2 + \cdots + f_{s+2}^2 + f_{s+1}^2\right) \\ &\quad + \left(f_{s+r-3}^2 + f_{s+r-4}^2 + \cdots + f_{s+1}^2 + f_s^2\right) \\ &\quad + f_{s+r-1}^2 + f_{s+r-2}^2 + 2f_s^2. \end{aligned}$$

Thus,

$$\begin{aligned} trF^2_{G_{f_r}} &= 3\sum_{t=s}^{s+r-2} f_t^2 + f_{s+r-1}^2 \\ &= 3\left(\sum_{t=1}^{s+r-2} f_t^2 - \sum_{t=1}^{s-1} f_t^2\right) + f_{s+r-1}^2 \\ &= 3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2. \end{aligned}$$

(3) By using (1) and (2), we get

$$\begin{aligned} \sum_{t=1}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)^2 &= \sum_{t=1}^{r}\lambda_t^2 - 2\frac{trF_{G_{f_r}}}{r}\sum_{t=1}^{r}\lambda_t + \sum_{t=1}^{r}\left(\frac{trF_{G_{f_r}}}{r}\right)^2 \\ &= \sum_{t=1}^{r}\lambda_t^2 - 2\frac{\left(trF_{G_{f_r}}\right)^2}{r} + r\left(\frac{trF_{G_{f_r}}}{r}\right)^2 \\ &= 3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2 \\ &\quad - \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right)^2. \end{aligned}$$

□

Theorem 2.2. The following equalities hold true for the generalized Lucas-Frank matrix $F_{G_{l_r}}$

(1) $trF_{G_{l_r}} = l_{s+r+1} - l_{s+1}$,

(2) $trF^2_{G_{l_r}} = 3\left(l_{s+r-2}l_{s+r-1} - l_{s-1}l_s\right) + l_{s+r-1}^2$,

(3) $\sum_{t=1}^{r}\left(\mu_t - \frac{trF_{G_{l_r}}}{r}\right)^2 = 3\left(l_{s+r-2}l_{s+r-1} - l_{s-1}l_s\right) + l_{s+r-1}^2 - \frac{1}{r}\left(l_{s+r+1} - l_{s+1}\right)^2$,

where μ_t's $(t = 1, 2, \ldots, r)$ are the eigenvalues of the matrix $F_{G_{l_r}}$.

Proof. The proof follows a similar approach to that of Theorem 2.1. □

Theorem 2.3. Let the eigenvalues of the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$ are ordered as $\lambda_1 > \lambda_2 > \ldots > \lambda_r$. Then, the largest eigenvalue λ_1 has the following upper bound

$$\begin{aligned} \lambda_1 &\le \left(1 - \frac{1}{r}\right)^{\frac{1}{2}} \left(3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2 \right. \\ &\qquad \left. - \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right)^2\right)^{\frac{1}{2}} + \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right). \end{aligned}$$

Proof. The equality

$$\lambda_1 - \frac{trF_{G_{f_r}}}{r} = -\sum_{t=2}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)$$

holds true for the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$. Then, we have

$$|\lambda_1 - \frac{trF_{G_{f_r}}}{r}| \leq \sum_{t=2}^{r}|\lambda_t - \frac{trF_{G_{f_r}}}{r}|. \tag{2.4}$$

By means of [16], we have the inequality

$$\sum_{t=1}^{r} o_t \sum_{t=1}^{r} o_t a_t b_t \geq \sum_{t=1}^{r} o_t a_t \sum_{t=1}^{r} o_t b_t, \tag{2.5}$$

where $o = (o_t)$ is a positive real number sequence and $a = (a_t)$ and $b = (b_t)$, $(t = 1, 2, \ldots, r)$ are non-negative real number sequences with same monocity. Moreover, if the sequences $a = (a_t)$ and $b = (b_t)$ have opposite monocity, then the inequality (2.5) reverses [16]. If the inequality (2.5) is used for the right hand side of the inequality (2.4) with $a_t = \dfrac{1}{|\lambda_t - \frac{trF_{G_{f_r}}}{r}|}$ and $b_t = o_t = |\lambda_t - \dfrac{trF_{G_{f_r}}}{r}|$, we have

$$|\lambda_1 - \frac{trF_{G_{f_r}}}{r}| \leq \sum_{t=2}^{r}|\lambda_t - \frac{trF_{G_{f_r}}}{r}| \leq \sqrt{(r-1)\sum_{t=2}^{r}|\lambda_t - \frac{trF_{G_{f_r}}}{r}|^2}\ .$$

Then,

$$\left(\lambda_1 - \frac{trF_{G_{f_r}}}{r}\right)^2 \leq (r-1)\left(\sum_{t=1}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)^2 - \left(\lambda_1 - \frac{trF_{G_{f_r}}}{r}\right)^2\right),$$

$$\frac{r}{r-1}\left(\lambda_1 - \frac{trF_{G_{f_r}}}{r}\right)^2 \leq \sum_{t=1}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)^2,$$

$$\lambda_1 - \frac{trF_{G_{f_r}}}{r} \leq \sqrt{\left(1-\frac{1}{r}\right)\sum_{t=1}^{r}\left(\lambda_t - \frac{trF_{G_{f_r}}}{r}\right)^2}.$$

Considering Theorem 2.1 (3)

$$\lambda_1 \leq \left(1-\frac{1}{r}\right)^{\frac{1}{2}}\left(3\left(f_{s+r-2}f_{s+r-1} - f_{s-1}f_s\right) + f_{s+r-1}^2 - \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right)^2\right)^{\frac{1}{2}} + \frac{1}{r}\left(f_{s+r+1} - f_{s+1}\right).$$

□

Theorem 2.4. Let the eigenvalues of the generalized Lucas-Frank matrix $F_{G_{l_r}}$ are ordered as $\mu_1 > \mu_2 > \ldots > \mu_r$. Then, there is the following upper bound for the largest eigenvalue μ_1

$$\begin{aligned}\mu_1 \quad &\leq \left(1-\frac{1}{r}\right)^{\frac{1}{2}} \left(3\left(l_{s+r-2}l_{s+r-1}-l_{s-1}l_s\right)+l_{s+r-1}^2\right.\\ &\qquad \left.-\frac{1}{r}\left(l_{s+r+1}-l_{s+1}\right)^2\right)^{\frac{1}{2}}+\frac{1}{r}\left(l_{s+r+1}-l_{s+1}\right).\end{aligned}$$

Proof. The proof follows a similar approach to that of Theorem 2.3. □

The next section includes the Euclidean and spectral norms of the generalized Fibonacci-Frank and Lucas-Frank matrices.

3 Some norms of the generalized Fibonacci-Frank and Lucas-Frank matrices

The Euclidean (Frobenius) and spectral norm of an $m \times n$ matrix X are

$$\|X\|_F = \left[\sum_{u=1}^{m}\sum_{v=1}^{n} \mid x_{uv} \mid^2\right]^{\frac{1}{2}} \quad \text{and} \quad \|X\|_2 = \sqrt{\max_{1\leq t\leq n} \lambda_t\left(X^H X\right)},$$

respectively, where X^H is the conjugate transpose of X and λ_t's are the eigenvalues of $X^H X$ [17].

The maximum row length norm $r_1(X)$ and the maximum column length norm $c_1(X)$ of the $m \times n$ matrix $X = [x_{uv}]$ are

$$r_1(X) = \max_u \sqrt{\sum_v \mid x_{uv} \mid^2} \quad \text{and} \quad c_1(X) = \max_v \sqrt{\sum_u \mid x_{uv} \mid^2},$$

respectively [17].

Let the $m \times n$ matrices $X = [x_{uv}]$, $Y = [y_{uv}]$, and $Z = [z_{uv}]$ have the relation $Y \circ Z = X$, then

$$\|X\|_2 \leq r_1(Y)\, c_1(Z),$$

where $Y \circ Z = [y_{uv} z_{uv}]$ is the Hadamard product of the matrices Y and Z [17].

Theorem 3.1. The Euclidean norm of the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$ is

$$\begin{aligned}\|F_{G_{f_r}}\|_F &= \left(\frac{r+2}{5}\left(l_{2s+2r-3}-l_{2s-1}\right)\right.\\ &\quad -\frac{1}{5}\left(l_{2s-2}-rl_{2s+2r-4}+(r-1)\,l_{2s+2r-2}\right)\\ &\quad\left. -\frac{1}{2}(-1)^{r+s}-\frac{2r+3}{10}(-1)^{s}+f_{s+r-1}^{2}\right)^{\frac{1}{2}}.\end{aligned}$$

Proof. By considering the Binet formulas for the Fibonacci and Lucas sequences, we have

$$\begin{aligned}\|F_{G_{f_r}}\|_F^2 &= \sum_{t=1}^{r-1}(r-t+2)\,f_{s+t-1}^2+f_{s+r-1}^2\\ &= \sum_{t=1}^{r-1}(r+2)\,f_{s+t-1}^2-\sum_{t=1}^{r-1}tf_{s+t-1}^2+f_{s+r-1}^2\\ &= \frac{(r+2)}{5}\sum_{t=1}^{r-1}\left(\varphi^{s+t-1}-\psi^{s+t-1}\right)^2\\ &\quad -\frac{1}{5}\sum_{t=1}^{r-1}t\left(\varphi^{s+t-1}-\psi^{s+t-1}\right)^2+f_{s+r-1}^2\\ &= \frac{r+2}{5}\sum_{t=1}^{r-1}\left(\varphi^{2s-2}\left(\varphi^{t}\right)^2+\psi^{2s-2}\left(\psi^{t}\right)^2-2\,(-1)^{s+t-1}\right)\\ &\quad -\frac{1}{5}\sum_{t=1}^{r-1}t\left(\varphi^{2s-2}\left(\varphi^{t}\right)^2+\psi^{2s-2}\left(\psi^{t}\right)^2-2\,(-1)^{s+t-1}\right)+f_{s+r-1}^2.\end{aligned}$$

By using the well-known equalities $\sum_{t=1}^{r-1} m^t = \frac{m^r-m}{m-1}$ and

$\sum_{t=1}^{r-1} tm^t = \frac{m - rm^r + (r-1)m^{r+1}}{(m-1)^2}$ for an arbitrary m, $(m \neq 1)$, we get

$$\begin{aligned}
\|F_{G_{f_r}}\|_F^2 &= \frac{r+2}{5}\left(\varphi^{2s-2}\frac{(\varphi^2)^r - \varphi^2}{\varphi^2 - 1} + \psi^{2s-2}\frac{(\psi^2)^r - \psi^2}{\psi^2 - 1} - 2(-1)^{s-1}\sum_{t=1}^{r-1}(-1)^t\right) \\
&\quad -\frac{1}{5}\left(\varphi^{2s-2}\left(\frac{\varphi^2 - r(\varphi^2)^r + (r-1)(\varphi^2)^{r+1}}{(\varphi^2-1)^2}\right)\right. \\
&\quad +\psi^{2s-2}\left(\frac{\psi^2 - r(\psi^2)^r + (r-1)(\psi^2)^{r+1}}{(\psi^2-1)^2}\right) \\
&\quad \left. -2(-1)^{s-1}\sum_{t=1}^{r-1} t(-1)^t\right) + f_{s+r-1}^2 \\
&= \frac{r+2}{5}\left(\left(\varphi^{2s+2r-3} + \psi^{2s+2r-3}\right) - \left(\varphi^{2s-1} + \psi^{2s-1}\right)\right. \\
&\quad \left. -(-1)^{s-1}((-1)^r + 1)\right) - \frac{1}{5}\left(\left(\varphi^{2s-2} + \psi^{2s-2}\right)\right. \\
&\quad -r\left(\varphi^{2s+2r-4} + \psi^{2s+2r-4}\right) + (r-1)\left(\varphi^{2s+2r-2} + \psi^{2s+2r-2}\right) \\
&\quad \left. +(-1)^{s-1}\frac{1 + (2r-1)(-1)^r}{2}\right) + f_{s+r-1}^2 \\
&= \frac{r+2}{5}\left(l_{2s+2r-3} - l_{2s-1}\right) - \frac{1}{5}\left(l_{2s-2} - rl_{2s+2r-4}\right. \\
&\quad \left. +(r-1)l_{2s+2r-2}\right) - \frac{1}{2}(-1)^{r+s} - \frac{2r+3}{10}(-1)^s + f_{s+r-1}^2.
\end{aligned}$$

□

Theorem 3.2. The Euclidean norm of the generalized Lucas-Frank matrix $F_{G_{l_r}}$ is

$$\begin{aligned}
\|F_{G_{l_r}}\|_F &= \left((r+2)\left(l_{2s+2r-3} - l_{2s-1}\right) - l_{2s-2} + rl_{2s+2r-4}\right. \\
&\quad -(r-1)l_{2s+2r-2} + \frac{5}{2}(-1)^{s+r} \\
&\quad \left. +\frac{2r+3}{2}(-1)^s + l_{s+r-1}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof. The proof follows a similar approach to that of Theorem 3.1.

□

Theorem 3.3. The spectral norm of the generalized Fibonacci-Frank matrix $F_{G_{f_r}}$ has the inequality

$$\|F_{G_{f_r}}\|_2 \leq \sqrt{\left(f_{s+r-1}^2 + 1\right)\left(f_{s+r-2}^2 + r - 1\right)}.$$

Proof. By using the Hadamard product, the matrix $F_{G_{f_r}}$ can be written as

$F_{G_{f_r}} = X \circ Y$, where

$$X = [x_{uv}]_{u,v=1}^{r} = \begin{bmatrix} f_{s+r+1} & 1 & 0 & 0 & \dots & 0 & 0 \\ f_{s+r-2} & f_{s+r-2} & 1 & 0 & \dots & 0 & 0 \\ f_{s+r-3} & f_{s+r-3} & f_{s+r-3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+1} & f_{s+1} & f_{s+1} & f_{s+1} & \dots & f_{s+1} & 1 \\ f_s & f_s & f_s & f_s & \dots & f_s & f_s \end{bmatrix}$$

and

$$Y = [y_{uv}]_{u,v=1}^{r} = \begin{bmatrix} 1 & f_{s+r-2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & f_{s+r-3} & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & f_{s+r-4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & f_s \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

It is easy to see by the induction method that the inequality $2f_{s+r-2} < f_{s+r+1}$ holds true for the Fibonacci number sequence. Then, the maximum row length norm of X and maximum column length norm of Y are

$$r_1(X) = \max_u \sqrt{\sum_v |x_{uv}|^2} = \sqrt{f_{s+r-1}^2 + 1},$$

$$c_1(Y) = \max_v \sqrt{\sum_u |y_{uv}|^2} = \sqrt{f_{s+r-2}^2 + r - 1}$$

respectively. Thus, we have

$$\|F_{G_{f_r}}\|_2 \leq r_1(X)\, c_1(Y) = \sqrt{\left(f_{s+r-1}^2 + 1\right)\left(f_{s+r-2}^2 + r - 1\right)}$$

as desired. □

Theorem 3.4. The spectral norm of the generalized Lucas-Frank matrix $F_{G_{l_r}}$ has the inequality

$$\|F_{G_{l_r}}\|_2 \leq \sqrt{\left(l_{s+r-1}^2 + 1\right)\left(l_{s+r-2}^2 + r - 1\right)}.$$

Proof. The proof follows a similar approach to that of Theorem 3.3. □

Finally, we provide some examples to illustrate our results.

4 Examples

Tables 1 and 2 include some examples for the upper bounds for the largest eigenvalues of the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$, respectively.

r	s	Largest Eigenvalue	Upper bound
3	4	12.5208	12.6666
	6	32.7797	33.1833
	8	85.8189	86.8840
5	4	33.1074	34.3639
	6	86.6690	89.9671
	8	226.8998	235.5376
7	4	86.6716	90.7103
	6	226.9066	237.4785
	8	594.0480	621.7254

Table 1: *Some upper bounds for the largest eigenvalue of the matrix $F_{G_{f_r}}$ for the values $r = 3, 5, 7$ and $s = 4, 6, 8$ according to Theorem 2.3.*

r	s	Largest Eigenvalue	Upper bound
3	4	28	28.3707
	6	73.2987	74.2182
	8	191.8972	194.2854
5	4	74.0162	76.8441
	6	193.7925	201.1741
	8	507.3613	526.6787
7	4	193.7982	202.8269
	6	507.3764	531.0153
	8	1328.3309	1390.2193

Table 2: *Some upper bounds for the largest eigenvalue of the matrix $F_{G_{l_r}}$ for the values $r = 3, 5, 7$ and $s = 4, 6, 8$ according to Theorem 2.4.*

The Euclidean norms of the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$ for $r = 3$ and $s = 4$ are obtained as follows according to Theorems 3.1 and 3.2

$$\begin{aligned} \|F_{G_{f_3}}\|_F &= \left(l_{11} - l_7 - \frac{1}{5}\left(l_6 - 3l_{10} + 2l_{12}\right) + \frac{1}{2} - \frac{9}{10} + f_6^2\right)^{\frac{1}{2}} \\ &= 13.2288 \end{aligned}$$

and

$$\begin{aligned}\|F_{G_{l_3}}\|_F &= \left(5\left(l_{11}-l_7\right)-l_6+3l_{10}-2l_{12}-\frac{5}{2}+\frac{9}{2}+l_6^2\right)^{\frac{1}{2}}\\ &= 29.7153.\end{aligned}$$

Tables 3 and 4 include some examples for the upper bounds for the spectral norms of the matrices $F_{G_{f_r}}$ and $F_{G_{l_r}}$, respectively.

r	s	Spectral norm	Upper bound
3	4	12.7614	41.8927
	6	33.4372	274.9218
	8	87.5508	1871.9263
5	4	34.9246	276.5249
	6	91.4455	1873.5421
	8	239.4121	12819.5446
7	4	92.0298	1875.1565
	6	240.9420	12821.1622
	8	630.7961	87846.1630

Table 3: *Some upper bounds for the spectral norm of the matrix $F_{G_{f_r}}$ for the values $r = 3, 5, 7$ and $s = 4, 6, 8$ according to Theorem 3.3.*

r	s	Spectral norm	Upper bound
3	4	28.5934	199.9375
	6	74.7902	1364.9286
	8	195.7780	9349.9273
5	4	78.1188	1366.5467
	6	204.4879	9351.5453
	8	535.3454	64081.5451
7	4	205.5950	9353.1631
	6	538.7664	64083.1631
	8	1410.5045	439208.1631

Table 4: *Some upper bounds for the spectral norm of the matrix $F_{G_{l_r}}$ for the values $r = 3, 5, 7$ and $s = 4, 6, 8$ according to Theorem 3.4.*

According to these examples, both the largest eigenvalue and spectral norm bounds of the generalized Fibonacci-Frank and Lucas-Frank matrices are closer to their real values of these concepts for smaller values of n and k than for larger values.

Conclusion

In this paper, we have introduced new generalizations of the Frank matrix, which is a special maximum matrix that attracts attention with its eigenvalues in the literature. We have called these generalizations the generalized Fibonacci-Frank and Lucas-Frank matrices since their entries were selected starting from the sth Fibonacci and Lucas numbers, respectively. We have obtained the upper bounds for the largest eigenvalues of these matrices and investigated their Euclidean and spectral norms. Additionally, we have given some numerical examples to illustrate our results.

References

[1] Frank W.L. (1958). Computing eigenvalues of complex matrices by determinant evaluation and by methods of Danilewski and Wielandt. Journal of the Society for Industrial and Applied Mathematics, 6, 378-392.

[2] Hake J.-F. (1985). A remark on Frank matrices. Computing (Wien. Print) 35, 375-379.

[3] Varah J.M. (1986). A generalization of the Frank matrix. SIAM Journal on Scientific and Statistical Computing 7(3), 835-839. https://doi.org/10.1137/0907056

[4] Mersin E.Ö., Bahşi M. & Maden A.D. (2020). Some properties of generalized Frank matrices. Mathematical Sciences and Applications E-Notes 8(2), 170-177. https://doi.org/10.36753/mathenot.672621

[5] Mersin E.Ö.& Bahşi M. (2021). Sturm theorem for the generalized Frank matrix. Hacettepe Journal of Mathematics and Statistics 50(4), 1002–1011. https://doi.org/10.15672/hujms.773281

[6] Polatlı E. (2023). On some properties of a generalized min matrix. AIMS Mathematics, 8(11), 26199-26212.

[7] Koshy T. (2001). Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs, and Tracts, New York: Wiley.

[8] Vajda S. (1989). Fibonacci and Lucas Numbers, and the Golden Section. John Wiley and Sons, New York.

[9] Bahşi M. (2015). On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers. TWMS Journal of Pure and Applied Mathematics, 6(1), 84-92.

[10] He C., Ma J., Zhang K.& Wang Z. (2015). The upper bound estimation on the spectral norm r-circulant matrices with the Fibonacci and Lucas numbers. Journal of Inequalities and Applications, 2015(1), 1-10.

[11] Jafari-Petroudi S. & Pirouz M. (2016). On the bounds for the spectral norm of particular matrices with Fibonacci and Lucas numbers. International Journal of Advances in Applied Mathematics and Mechanics 3, 82-90.

[12] Solak, S. (2005). On the norms of circulant matrices with the Fibonacci and Lucas numbers. Applied Mathematics and Computation, 160(1), 125-132.

[13] Tuglu, N., & Kızılateş, C. (2015). On the Norms of Some Special MatricesWith the Harmonic Fibonacci Numbers. Gazi University Journal of Science, 28(3), 497-501.

[14] Mersin E. Ö. (2021). Properties of generalized Frank matrices, PhD. Thesis. Aksaray University Graduate School of Natural and Applied Sciences, Department of Mathematics.

[15] Mersin, E. Ö., & Bahşi, M. (2024). Bounds for the maximum eigenvalues of the Fibonacci-Frank and Lucas-Frank matrices. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 73(2), 420-436.

[16] Milovanovic, I. Z., Milovanovic, E. I., & Matejic, M. (2018). Some inequalities for general sum–connectivity index. MATCH Commun. Math. Comput. Chem, 79, 477-489.

[17] Johnson, C. R., & Horn, R. A. (1985). Matrix analysis. Cambridge: Cambridge University Press.

3. ON SOME POLYGONAL NUMBERS

Aslı Özen **Ahmet Emin**

Abstract

Polygonal numbers are integer sequences associated with regular geometric shapes. Polygonal numbers are also called n-gonal numbers because they are related to the number of sides of regular polygons. Triangular numbers represent points arranged along the edges of a triangle; square numbers represent points arranged in a square; pentagonal numbers represent points arranged in a pentagon, and so on. This study will discuss some identities defined for triangular and square numbers and the visual proofs of these identities.

Keywords. Figurate numbers, Polygonal numbers, Triangular numbers, Square numbers, Visual proofs

ORCID: 0009-0004-3095-4232
e-mail: 2228139304@ogrenci.karabuk.edu.tr

ORCID: 0000-0001-7791-7181
e-mail: ahmetemin@karabuk.edu.tr

1 Introduction

Figurate numbers are positive integers that can be represented geometrically by an arrangement of dots or physically by an arrangement of objects such as pebbles. Figurate numbers are among the most widely studied topics in number theory due to their role in connecting geometry and number theory.

Points representing figurate numbers are given different names based on the shapes they form in space or on the plane. If the points are arranged in a regular polygon shape on the plane, the figurate numbers are called *Polygonal Numbers*. If the points are clustered in a regular polygon around a point on the plane, the figurate numbers are called *Centered Polygonal Numbers*. Additionally, if the points are formed as a cube in three-dimensional space, the figurate numbers are called *Cubic Numbers*. If they are formed in the shape of a pyramid with an equilateral triangle base, the figurate numbers are called *Regular Tetrahedral*. Also, pronic, trapezoidal, and polygram numbers are examples of other two-dimensional figurate numbers. For further information on figurate numbers, see [1, 2, 3].

In this study, some polygonal numbers and their properties will be discussed. Also, visual proofs of some identities will be given.

2 Polygonal Numbers

A polygonal number is a sequence of numbers obtained by arranging pebbles, balls, or dots representing numbers as a regular polygon on a plane. Polygonal numbers are named according to the number of sides of the regular polygons formed on the plane. Accordingly, we start with a fixed point in the plane. Two points are added to this fixed point, and the resulting three points are arranged as an equilateral triangle. Three more points are added to this three-point equilateral triangle, and one more line is added to the existing triangle. A larger equilateral triangle is obtained if these six points are arranged in the form of an equilateral triangle so that the points in each line are equidistant. Four more points are added to this six-point equilateral triangle, and one more line is added to the existing triangle. If these ten points are arranged as an equilateral triangle with the points in each line being equidistant, an equilateral triangle of a larger size is obtained with ten points. Thus, this process continues , and a larger equilateral triangle is created each time.

As a result, if two, three, four, five, ... points are added to a fixed point at the beginning, and these points are arranged in the form of an equilateral triangle, an equilateral triangle of larger size is obtained each time. The number of points in this equilateral triangle is 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... respectively, and these numbers form a sequence of positive integers. The terms in this sequence of numbers are called *Triangular Numbers*, see [1].

Similarly, we start with a fixed point on the plane. Three more points are added to this fixed point, and the resulting four points are arranged in a square. Five more points are added to this four-point square, forming a larger square when one more row is added. A larger square is formed if these nine points are arranged in a square pattern, with the points in each

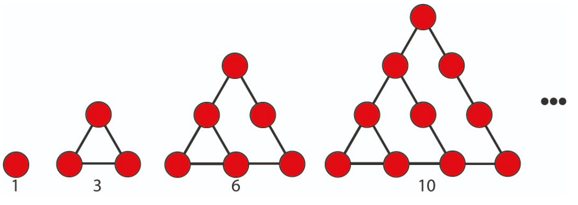


Figure 1: *The first four triangular numbers*

row equidistant. Seven more points are added to this nine-point square, and one more row is added to the existing square. If these sixteen points are arranged in a square form, with the points in each row equidistant, a larger square is obtained with sixteen points. Thus, this process continues indefinitely, and a larger square is created each time.

As a result, if three, five, seven, nine, ... points are added to a fixed point at the beginning, and these points are arranged in a square form, a larger square is obtained each time. The number of points in these squares is 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ... respectively, forming a sequence of positive integers. The terms in this sequence are called *Square Numbers*; see [1].

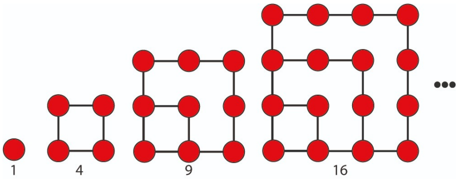


Figure 2: *The first four square numbers*

If four, seven, ten, thirteen, ... points are added to a fixed point, and these points are arranged in the plane as a regular pentagon, a larger regular pentagon is obtained each time. The number of points in these regular pentagons is 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ... respectively, forming a sequence of positive integers. The terms in this sequence are called *Pentagonal Numbers*.

If the same procedure is continued, *Hexagonal Numbers* can be constructed. Thus, the terms of the sequence of hexagonal numbers are 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, ... respectively. Similarly, 1, 7, 18, 34, 55, 81, 112, 148, 189, 235, ... are the sequence of *heptagonal numbers*, and 1, 8, 21, 40, 65, 96, 133, 176, 225, 280, ... are the first few terms of the sequence of *octagonal numbers* (see [1]).

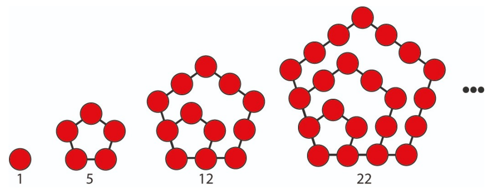


Figure 3: *The first four pentagonal numbers*

Thus, we have constructed some subclasses of polygonal numbers. This study will focus only on triangular and square numbers and their properties.

3 Triangular and Square Numbers

Let the sequence of triangular numbers be denoted by Tn. As seen from the construction of the triangular numbers above, we initially had one point. Therefore, $T_1 = 1$. Also, $T_2 = T_1 + 2$, $T_3 = T_2 + 3$, $T_4 = T_3 + 4$, $T_5 = T_4 + 5$, and so on to obtain all terms of the sequence. Thus, a recursive relation between triangular numbers is of the form $T_n = T_{n-1} + n$ for $n \geq 1$ with $T_0 = 0$ (see [2]). From here, we can derive the expression that gives the general term for the triangular numbers. Accordingly, the sum of

$$T_n = 1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2} \tag{3.1}$$

for $n \geq 1$ expresses the general term of the triangular numbers sequence.

Let the sequence of square numbers be denoted by S_n. As seen from the construction of square numbers above, we initially had one point. Therefore, $S_1 = 1$ and $S_2 = S_1 + 3,\ S_3 = S_2 + 5,\ S_4 = S_3 + 7,\ S_5 = S_4 + 9$, and so on to obtain all terms of the sequence. Then, a recursive relation for square numbers is of the form

$$S_n = S_{n-1} + (2n-1)\,, \quad S_0 = 0 \tag{3.2}$$

for $n \geq 1$. We can obtain the general term for square numbers from here. Accordingly, for $n \geq 1$, the sum

$$S_n = 1 + 3 + 5 + \ldots + (2n-3) + (2n-1) = n^2 \tag{3.3}$$

represents the general term of the sequence of square numbers.

Some identities obtained regarding triangular and square numbers will be presented in a new section below.

4 Identities Involving Triangular and Square Numbers

In this section, we will present some interesting identities involving triangular and square numbers. Also, an alternative proof method, visual proof, will be used to prove these identities. For further information on visual proof, see [3, 4, 5, 6, 8, 7, 9].

Theorem 4.1. Let T_n be the n^{th} triangular number. The representation of the n^{th} triangular number in terms of binomial coefficients is obtained using the identity

$$T_n = \binom{n+1}{2} \tag{4.2}$$

for $n \geq 1$.

Proof. The coefficients of the terms in the binomial expansion are obtained with the identity

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \tag{4.3}$$

for $n \geq r \geq 1$. So, from the identities 4.3, we have that

$$\binom{n+1}{2} = \frac{(n+1)!}{(n-1)!.2!} = \frac{(n+1)\,.n.\,(n-1)!}{(n-1)!.2!} = \frac{n\,(n+1)}{2} = T_n.$$

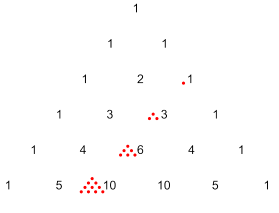


Figure 4: *Binomial coefficients*

□

Theorem 4.4. The sum of two consecutive triangular numbers is a square number. That is, Let T_{n-1} and T_n be two consecutive triangular numbers and let S_n be a square number for $n \geq 1$. So,

$$T_{n-1} + T_n = S_n. \tag{4.5}$$

Proof. From the general terms of the sequence of triangular and square numbers, we find that

$$T_{n-1}+T_n=\frac{(n-1)\,n}{2}+\frac{n\,(n+1)}{2}$$
$$=\frac{n^2-n+n+n^2}{2}=\frac{2n^2}{2}=n^2=S_n$$

for $n \geq 1$.

A visual proof can serve as an alternative to an algebraic proof. For instance, when $n=6$, you can see that the sum of the fifth triangular number T_5 (red triangle) and the sixth triangular number T_6 (blue triangle) equals the sixth square number S_6.

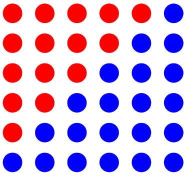


Figure 5: $T_{n-1}+T_n=S_n$

□

The following theorem presents an identity for the double-indexed terms within the triangular numbers.

Theorem 4.6 ([1]). Let T_n be a triangular number. Then,

$$T_{2n}=3T_n+T_{n-1}$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we find that

$$3T_n+T_{n-1}=3\frac{n\,(n+1)}{2}+\frac{(n-1)\,n}{2}$$
$$=\frac{3n^2+3n+n^2-n}{2}$$
$$=\frac{4n^2+2n}{2}=\frac{2n\,(2n+1)}{2}=T_{2n}$$

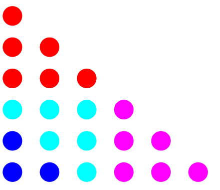


Figure 6: $T_{2n} = 3T_n + T_{n-1}$

for $n \geq 1$. In particular, if we take $n = 3$, the following figure can be seen as a proof model for this identity.

□

Similarly, there is an identity for the odd-indexed terms of the triangular numbers, as stated in the following theorem.

Theorem 4.7 ([1]). Let T_n be a triangular number. Then,

$$T_{2n+1} = 3T_n + T_{n+1}$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we get that

$$\begin{aligned} 3T_n + T_{n+1} &= 3\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\ &= \frac{3n^2 + 3n + n^2 + 2n + n + 2}{2} \\ &= \frac{4n^2 + 6n + 2}{2} = \frac{(2n+1)(2n+2)}{2} = T_{2n+1} \end{aligned}$$

for $n \geq 1$. In particular, if we take $n = 2$, the following figure can be seen as a proof model for this identity.

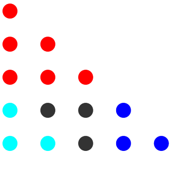


Figure 7: $T_{2n+1} = 3T_n + T_{n+1}$

□

The next theorem is more commonly known as Plutarch's identity, though it is also referred to as Diophantus's identity.

Theorem 4.8 ([1, 2]). Let T_n be a triangular number, S_n be a square number. Then,

$$S_{2n+1} = 8T_n + 1$$

for $n \geq 1$.

Proof. From the general terms of the sequences of triangular and square numbers, we find that

$$\begin{aligned} 8T_n + 1 &= 8\frac{n(n+1)}{2} + 1 \\ &= \frac{8n^2 + 8n}{2} + 1 \\ &= 4n^2 + 4n + 1 \\ &= (2n+1)^2 = S_{2n+1} \end{aligned}$$

for $n \geq 1$. In particular, if we take $n = 10$, the following figure can be seen as a proof model for this identity.

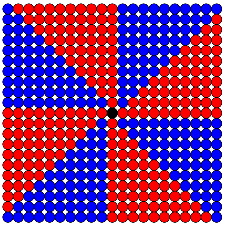


Figure 8: $S_{2n+1} = 8T_n + 1$

□

The identity between two consecutive even or odd indexed triangular numbers is given by the following theorem.

Theorem 4.9 ([1]). Let T_n be a triangular number. Then we get that

$$2T_n + 1 = T_{n-1} + T_{n+1}$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we find that

$$\begin{aligned}2T_n + 1 = 2\frac{n(n+1)}{2} + 1 &= \frac{2n^2 + 2n}{2} + 1 \\ &= \frac{n^2 + n^2 + 3n - n}{2} + 1 \\ &= \frac{n^2 - n}{2} + \frac{n^2 + 3n + 2}{2} \\ &= \frac{(n-1)n}{2} + \frac{(n+1)(n+2)}{2} \\ &= T_{n-1} + T_{n+1}\end{aligned}$$

for $n \geq 1$. In particular, if we take $n = 3$, the following figure can be seen as a proof model for this identity.

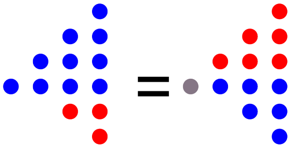


Figure 9: $2T_n + 1 = T_{n-1} + T_{n+1}$

□

The following theorem gives the identity that expresses the difference between two consecutive triangular numbers.

Theorem 4.10. Let T_n be a triangular number. Then we get that

$$T_n - T_{n-1} = n$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we get that

$$T_n - T_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = \frac{n^2 + n - n^2 + n}{2} = \frac{2n}{2} = n$$

for $n \geq 1$. □

The following theorem provides the identity that expresses the sum of the squares of two consecutive triangular numbers

Theorem 4.11 ([3]). Let T_n be a triangular number. Then we get that

$$(T_{n-1})^2 + (T_n)^2 = T_{n^2}$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we find that

$$\begin{aligned}(T_{n-1})^2 + (T_n)^2 &= \left[\frac{(n-1)\,n}{2}\right]^2 + \left[\frac{n\,(n+1)}{2}\right]^2 \\ &= \frac{(n-1)^2 n^2}{4} + \frac{n^2 (n+1)^2}{4} \\ &= \frac{n^2\left(n^2 - 2n + 1 + n^2 + 2n + 1\right)}{4} \\ &= \frac{n^2\left(n^2+1\right)}{2} = T_{n^2}\end{aligned}$$

for $n \geq 1$. In particular, if we take $n = 4$, the following figure can be seen as a proof model for this identity.

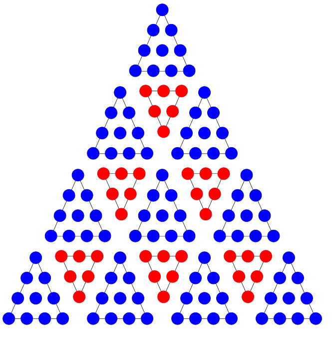


Figure 10: $(T_{n-1})^2 + (T_n)^2 = T_{n^2}$

Here, for $n = 4$, we present ${T_4}^2$ by using T_4 copies of T_4, and in the same way for ${T_3}^2$.

□

The following theorem provides the identity that relates two triangular numbers with indices m and n to the triangular number with index $m + n$, where m and n are any two natural numbers.

Theorem 4.12 ([1, 2, 4]). Let T_n and T_m be any two triangular numbers. Then, we have

$$T_{m+n} = T_m + T_n + mn$$

for $m, n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we have that

$$\begin{aligned} T_{m+n} &= \frac{(m+n)(m+n+1)}{2} \\ &= \frac{m^2 + m + n^2 + n + 2mn}{2} \\ &= \frac{m(m+1)}{2} + \frac{n(n+1)}{2} + mn \\ &= T_m + T_n + mn \end{aligned}$$

for $m, n \geq 1$. In particular, If we take $n = 3$ and $m = 4$, the following figure can be seen as a proof model for this identity

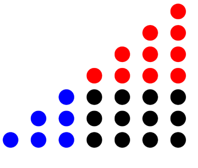


Figure 11: $T_{m+n} = T_m + T_n + mn$

□

The following theorem provides the identity for the indexed terms of triangular numbers that are multiples of four.

Theorem 4.13 ([5]). Let T_n be any triangular number. Then, we have

$$T_{4n} = 4(T_{2n} - T_n) + n + T_{2n-1}$$

for $n \geq 1$.

Proof. From the general terms of the sequence of triangular numbers, we get that

$$
\begin{aligned}
4\left(T_{2n}-T_n\right)+n+T_{2n-1} &= 4\left(\frac{2n\left(2n+1\right)}{2}-\frac{n\left(n+1\right)}{2}\right) \\
&\quad +n+\frac{\left(2n-1\right)2n}{2} \\
&= 4n\left(2n+1\right)-2n\left(n+1\right)+n+\left(2n-1\right)n \\
&= 8n^2+4n-2n^2-2n+n+2n^2-n \\
&= 8n^2+2n=\frac{4n\left(4n+1\right)}{2}=T_{4n}
\end{aligned}
$$

for $n \geq 1$. In particular, if we take $n = 4$, the following figure can be seen as a proof model for this identity.

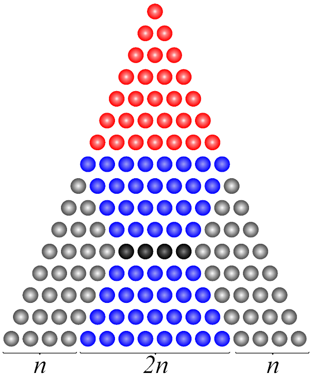


Figure 12: $T_{4n} = 4\left(T_{2n} - T_n\right) + n + T_{2n-1}$

□

In the next theorem, it is shown that the factors of a triangular number are related to a larger triangular number by the sum of the two triangular numbers formed by the factors. Since the proof of the theorem is given in detail in [6], the theorem will be given without proof.

Theorem 4.14 ([3, 6]). Let T_n be any triangular number, and let n, p and q be positive integers.Then, we have

$$T_n = pq \Leftrightarrow T_{n+p} + T_{n+q} = T_{n+p+q}.$$

In particular, if we take $n = 6, p = 3$ and $q = 7$, the following figure can be seen as a proof model for this identity.

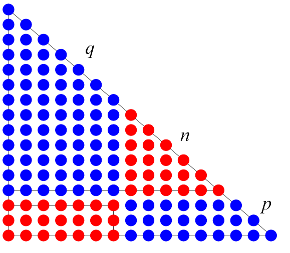


Figure 13: $T_{n+p+q} = T_{n+p} + T_{n+q} - T_n + pq$

5 Conclusion

In this study, we present identities related to polygonal numbers, which are a type of figurate number, and specifically to triangular and square numbers within the context of polygonal numbers. In addition to the known algebraic proofs, we also attempt to provide visual proofs of some identities. Our findings contribute to a deeper understanding of these mathematical concepts and demonstrate the power of visual methods in complementing algebraic proofs.

References

[1] Deza, E., & Deza, M. (2012). Figurate numbers. World Scientific.

[2] Garge, A. S., & Shirali, S. A. (2012). Triangular numbers. Resonance, 17, 672-681.

[3] Nelsen, R. B. (2018). Nuggets of number theory: a visual approach (Vol. 55). American Mathematical Soc..

[4] Edgar, T. (2020). Visual decompositions of polygonal numbers. The College Mathematics Journal, 51(1), 9-12.

[5] Caglayan, G. (2019). Proof Without Words: An Identity on Difference Between Triangular Numbers. Mathematics Magazine, 92(2), 107. https://doi.org/10.1080/0025570X.2019.1562297

[6] Nelsen, R. B. (2016). Multi-Polygonal Numbers. Mathematics Magazine, 89(3), 159-164.

[7] Nelsen, R. B. (1993). Proofs without words: Exercises in visual thinking (No. 1). MAA.

[8] Chakraborty, B. (2018). Proof without words: The sum of squares. Mathematical Intelligencer, 40(2), 20-20.

[9] Unal, H. (2010). A Visual Proof for the Sum of the First n Triangular Numbers. The Mathematical Intelligencer, 32, 6-7.

4. ON QUATERNIONS WITH JACOBSTHAL POLYNOMIAL COEFFICIENTS

Samet Arpacı **Fatih Yılmaz**

Abstract

In this study, we present a new family of polynomials which is called Jacobsthal quaternion polynomials. Then, we investigate some interesting properties of them.

Keywords. Jacobsthal numbers, Jacobsthal polynomials, Quaternions

ORCID: 0000-0001-7873-1979
e-mail: fatih.yilmaz@hbv.edu.tr
ORCID: 0000-0002-6646-8197
e-mail: samet.arpaci@hbv.edu.tr

1 Introduction

The well-known Jacobsthal number is defined by the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2$$

with initial conditions $J_0 = 0$ and $J_1 = 1$, see [1]. The Binet's formula of these sequences is characterized in the following form:

$$J_n = \frac{1}{3}(2^n - (-1)^n). \tag{1.1}$$

Please see some initial values in Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
J_n	0	1	1	3	5	11	21	43	85	171	341	683	1365

Table 5: *Some values of Jacobsthal numbers.*

The investigation of Jacobsthal polynomials is still a hot topic for many researchers. For example, in [2, 3], the authors give the Jacobsthal polynomials by the following recurrence relation:

$$J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x) \tag{1.2}$$

for $n \geq 2$ with initial conditions $J_0(x) = 0$ and $J_1(x) = 1$ and investigated some spectacular properties of the Jacobsthal polynomials such as summation formulas, Binet formulas, Simson formulas, and generating functions. In [4], the authors obtain convolutions for Jacobsthal type polynomials. For more details, please see the references [5, 6, 7, 8, 9, 10] and therein.

n	0	1	2	3	4	5	6
$J_n(x)$	0	1	1	$1+2x$	$1+4x$	$1+6x+4x^2$	$1+8x+12x^2$

Table 6: *Some values of Jacobsthal polynomial.*

A quaternion is four-dimensional hyper-complex number and is introduced by Sir William Rowan Hamilton, in 1843. These numbers have widespread applications in quantum physics, computer graphics, robotics and signal processing, etc. It has the form

$$q = q_0 + q_1 i + q_2 j + q_3 k = (q_0, q_1, q_2, q_3)$$

where q_0, q_1, q_2, q_3 are real numbers, and i, j, k are quaternionic units which satisfy the following equalities:

$$\begin{aligned} &i^2 = j^2 = k^2 = ijk = -1, \quad jk = i = -kj, \\ &ij = k = -ji, \quad ki = j = -ik. \end{aligned} \tag{1.3}$$

The set of all quaternions denoted by $\mathcal{H}$, is a non-commutative associative algebra over the real numbers. For a detailed information, see the references

[11, 12] and therein. In [13], Horadam introduced the Fibonacci and Lucas quaternions as

$$FQ_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$LQ_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

respectively, where F_n is the nth Fibonacci number, L_n is the nth Lucas number. There have been many studies in literature on Fibonacci and Lucas quaternions, see for example [14, 15, 16, 17, 18], among others.

The main aim of this paper is to define and study Jacobsthal quaternion polynomials. We shall give recurrence relations, Binet's formulas, generating functions and so on. In addition to these, we present summation formulas involving these type of quaternion polynomials.

2 On Quaternions with Jacobsthal Polynomials

In this section, we initially introduce Jacobsthal quaternion polynomials. Then, we obtain some amazing results for them.

Definition 2.1. For $n \geq 0$, $\{JQ_n(x)\}_{n=0}^{\infty}$ Jacobsthal quaternion polynomials are defined by

$$JQ_n(x) = (J_n(x), J_{n+1}(x), J_{n+2}(x), J_{n+3}(x)) \tag{2.2}$$

where $J_n(x)$ is the nth Jacobsthal polynomial. It is easy to see that the nth Jacobsthal quaternion polynomial is defined recursively by

$$JQ_n(x) = JQ_{n-1}(x) + 2xJQ_{n-2}(x), \; n \geq 2 \tag{2.3}$$

with the initial conditions

$$JQ_0(x) = (0, 1, 1, 1 + 2x) = i + j + (1 + 2x)k$$

and

$$JQ_1(x) = (1, 1, 1 + 2x, 1 + 4x) = 1 + i + (1 + 2x)j + (1 + 4x)k.$$

We give some initial values in Table (7).

n	0	1
$JQ_n(x)$	$i+j+(1+2x)k$	$1+i+(1+2x)j+(1+4x)k$

Table 7: *Some of Jacobsthal quaternion polynomials.*

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation $t^2-t-2x=0$ on the recurrence relation (2.3) of Jacobsthal quaternion polynomial. Here,

$$\alpha(x)=\frac{1-\sqrt{1+8x}}{2},\quad \beta(x)=\frac{1+\sqrt{1+8x}}{2}.$$

These roots satisfy the following properties:

- $\alpha(x)+\beta(x)=1,\ \ \alpha(x)-\beta(x)=\sqrt{1+8x},\ \ \alpha(x)\beta(x)=-2x.$
- $\frac{\alpha(x)}{\beta(x)}=-\frac{\alpha^2}{4x},\ \ \frac{\beta(x)}{\alpha(x)}=-\frac{\beta^2}{4x}.$

By the definition of Jacobsthal quaternion polynomials, we can write the following vector recurrence relation:

$$\begin{bmatrix} JQ_{n+1}(x) \\ JQ_n(x) \end{bmatrix} = Q(x)\begin{bmatrix} JQ_n(x) \\ JQ_{n-1}(x) \end{bmatrix}$$

where $Q(x)$ is the companion matrix of order 2, as follows:

$$Q(x)=\begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}.$$

Let us define a 2×2 matrix as below:

$$R(x)=\begin{bmatrix} JQ_2(x) & JQ_1(x) \\ JQ_1(x) & JQ_0(x) \end{bmatrix}.$$

Then, we have the following theorem.

Theorem 2.4. For $n\geq 1$, we have

$$Q^n(x)R(x)=\begin{bmatrix} JQ_{n+2}(x) & JQ_{n+1}(x) \\ JQ_{n+1}(x) & JQ_n(x) \end{bmatrix} \tag{2.5}$$

where JQ_n is nth Jacobsthal quaternion polynomial.

Proof. The proof can be seen easily exploiting the Mathematical Induction on n. □

Theorem 2.6. (Generating Function) The generating function for the Jacobsthal quaternion polynomials are given by

$$g(x,t)=\sum_{n=0}^{\infty} JQ_n(x)t^n=\frac{i+j+(1+2x)k+t(1+2xj+2xk)}{1-t-2xt^2}.$$

Proof. Taking into account the definition of the generating function and the Jacobsthal numbers, we can write the following equalities:

$$\begin{aligned}
g(x,t) &= JQ_0(x)+JQ_1(x)t+JQ_2(x)t^2+\ldots+JQ_n(x)t^n+\ldots\\
-tg(x,t) &= -tJQ_0(x)-JQ_1(x)t^2-JQ_2(x)t^3-\ldots-JQ_{n-1}(x)t^n-\ldots\\
-2xt^2g(x,t) &= -2xJQ_0(x)t^2-2xJQ_1(x)t^3-2xJQ_2(x)t^4-\ldots-2xJQ_{n-2}(x)t^n-\ldots
\end{aligned}$$

Then,

$$\begin{aligned}
(1-t-2xt^2)g(x,t) &= JQ_0(x)+t(JQ_1(x)-JQ_0(x))\\
&+t^2(JQ_2(x)-JQ_1(x)-2xJQ_0(x))\\
&+t^2(JQ_2(x)-JQ_1(x)-2xJQ_0(x))\\
&\vdots\\
&+t^n(JQ_n(x)-JQ_{n-1}(x)-2xJQ_{n-2}(x))+\ldots
\end{aligned}$$

Therefore,

$$\begin{aligned}
g(x,t) &= \frac{JQ_0(x)+t(JQ_1(x)-JQ_0(x))}{1-t-2xt^2}\\
&= \frac{i+j+(1+2x)k+t(1+2xj+2xk)}{1-t-2xt^2}
\end{aligned}$$

So, the proof is completed. □

Theorem 2.7. (Binet Formula) For $n\geq 0$, the Binet formula for the Jacobsthal quaternion polynomial is

$$JQ_n(x)=\frac{1}{\alpha-\beta}\left[\alpha^n A(x)+\beta^n B(x)\right] \tag{2.8}$$

where $\alpha=\frac{1-\sqrt{1+8x}}{2}$, $\beta=\frac{1+\sqrt{1+8x}}{2}$, $A(x)=(\alpha-1)JQ_0(x)+JQ_1(x)$ and $B(x)=(1-\beta)JQ_0(x)+JQ_1(x)$.

Proof. By using the generating function and the definition of the Jacobsthal quaternion polynomial, we obtain the following equalities

$$\begin{aligned} g(x,t) &= \frac{JQ_0(x) + t(JQ_1(x) - JQ_0(x))}{1 - t - 2xt^2} \\ &= \frac{A(x)}{1-\alpha} + \frac{B(x)}{1-\beta} \\ &= \sum_{n=0}^{\infty} A(x)\alpha^n t^n + \sum_{n=0}^{\infty} B(x)\beta^n t^n \\ &= \sum_{n=0}^{\infty} (\alpha^n A(x) + \beta^n B(x))t^n \end{aligned}$$

where $\alpha = \frac{1-\sqrt{1+8x}}{2}$, $\beta = \frac{1+\sqrt{1+8x}}{2}$. □

Theorem 2.9. (Cassini's Identity) For $n \geq 1$, we obtain

$$JQ_{n+1}(x)JQ_{n-1}(x) - JQ_n^2(x) = 2(-x)^{n-1}(JQ_2(x)JQ_0(x) - JQ_1^2(x))$$

where JQ_n is nth Jacobsthal quaternion polynomial.

Proof. It is obvious that

$$detR = JQ_2(x)JQ_0(x) - JQ_1^2(x)$$

and

$$detQ^{n-1} = 2(-x)^{n-1}.$$

Taking into account $detQ^{n-1}R$, we obtain the desired result. So, the proof is completed. □

Theorem 2.10. (Catalan's Identity) For $n, r \geq 0$, the Catalan Identity for the Jacobsthal quaternion polynomials are given by

$$JQ_{n-r}JQ_{n+r} - JQ_n^2 = \frac{\alpha^n\beta^n A(x)B(x)\left[(-1)^r\beta^{2r} + (-1)^r\alpha^{2r} - 2(4x)^r\right]}{(4x)^r\left(\alpha(x) - \beta(x)\right)^2}.$$

Proof. Using the Binet formula, given in (2.8), we get

$$\begin{aligned}
&JQ_{n-r}JQ_{n+r} - JQ_n^2 \\
&= \frac{[\alpha^{n-r}(x)A(x) + \beta^{n-r}(x)B(x)][\alpha^{n+r}(x)A(x) + \beta^{n+r}(x)B(x)]}{(\alpha(x) - \beta(x))^2} \\
&= \frac{\alpha^n(x)\beta^n(x)A(x)B(x)\left[\alpha^{-r}(x)\beta^r(x) + \beta^{-r}(x)\alpha^r(x) - 2\right]}{(\alpha(x) - \beta(x))^2} \\
&= \frac{\alpha^n(x)\beta^n(x)A(x)B(x)\left[\left(\frac{\beta(x)}{\alpha(x)}\right)^r + \left(\frac{\alpha(x)}{\beta(x)}\right)^r - 2\right]}{(\alpha(x) - \beta(x))^2} \\
&= \frac{\alpha^n(x)\beta^n(x)A(x)B(x)\left[\left(\frac{-\beta^2(x)}{4x}\right)^r + \left(\frac{-\alpha^2(x)}{4x}\right)^r - 2\right]}{(\alpha(x) - \beta(x))^2} \\
&= \frac{\alpha^n(x)\beta^n(x)A(x)B(x)\left[(-1)^r\beta^{2r}(x) + (-1)^r\alpha^{2r}(x) - 2(4x)^r\right]}{(4x)^r\left(\alpha(x) - \beta(x)\right)^2}.
\end{aligned}$$

□

Theorem 2.11. (Vajda Identity) For $n, m, r \geq 0$, the Vajda Identity is given by

$$\begin{aligned}
&JQ_{n+r}JQ_{n+k} - JQ_nJQ_{n+r+k} \\
&= \frac{1}{1+8x}\left[(\alpha^k(x) - \beta^k(x))(\beta^r(x) - \alpha^r(x))\right]\alpha^n(x)\beta^n(x)A(x)B(x).
\end{aligned}$$

Proof. From the Binet formula give in (2.8), we obtain

$$\begin{aligned}
&JQ_{n+r}JQ_{n+k} - JQ_nJQ_{n+r+k} \\
&= \frac{\left[\left(\alpha^{n+r}(x)A(x) + \beta^{n+r}(x)B(x)\right)\left(\alpha^{n+k}(x)A(x) + \beta^{n+k}(x)B(x)\right)\right]}{1+8x} \\
&- \frac{\left[(\alpha^n(x)A(x) + \beta^n(x)B(x))\left(\alpha^{n+r+k}(x)A(x) + \beta^{n+r+k}(x)B(x)\right)\right]}{1+8x} \\
&= \frac{\left[\alpha^{n+r}(x)\beta^{n+k}(x) + \beta^{n+r}\alpha^{n+k} - \alpha^n(x)\beta^{n+r+k} - \beta^n(x)\alpha^{n+r+k}A(x)B(x)\right]}{1+8x} \\
&= \frac{\left[\alpha^n(x)\beta^n(x)A(x)B(x)\left(\alpha^r(x)\beta^k(x) + \beta^r(x)\alpha^k(x) - \beta^{r+k}(x) - \alpha^{r+k}(x)\right)\right]}{1+8x} \\
&= \frac{\left[\alpha^n(x)\beta^n(x)A(x)B(x)\left(\alpha^r(x)\left(\beta^k(x) - \alpha^k(x)\right) + \beta^r(x)\left(\alpha^k(x) - \beta^k(x)\right)\right)\right]}{1+8x} \\
&= \frac{\left[\left(\alpha^k(x) - \beta^k(x)\right)(\beta^r(x) - \alpha^r(x))\right]\alpha^n(x)\beta^n(x)A(x)B(x)}{1+8x}.
\end{aligned}$$

So, the proof is completed. □

Theorem 2.12. (D'Ocagne Identity) For $n, m, r \geq 0$, the D'Ocagne identity is given by

$$JQ_m JQ_{n+1} - JQ_n JQ_{m+1} = \frac{1}{1+8x}\left[(\alpha-\beta)(\beta^m\alpha^n - \alpha^m\beta^n)\right] A(x)B(x).$$

Proof. From the Binet formula, given in (2.8), we obtain

$$\begin{aligned}
&JQ_m JQ_{n+1} - JQ_n JQ_{m+1}\\
&= \frac{\left[\left(\alpha^m(x)A(x)+\beta^m(x)B(x)\right)\left(\alpha^{n+1}(x)A(x)+\beta^{n+1}(x)B(x)\right)\right]}{1+8x}\\
&- \frac{\left[\left(\alpha^n(x)A(x)+\beta^n(x)B(x)\right)\left(\alpha^{m+1}(x)A(x)+\beta^{m+1}(x)B(x)\right)\right]}{1+8x}\\
&= \frac{\left[\alpha^m(x)\beta^{n+1}(x)+\beta^m\alpha^{n+1}-\alpha^n(x)\beta^{m+1}-\beta^n(x)\alpha^{m+1}A(x)B(x)\right]}{1+8x}\\
&= \frac{\left[\alpha^n(x)\beta^n(x)A(x)B(x)\left(\alpha^r(x)\beta^k(x)+\beta^r(x)\alpha^k(x)-\beta^{r+k}(x)-\alpha^{r+k}(x)\right)\right]}{1+8x}\\
&= \frac{\left[\left(\alpha^m(x)\left(\beta^{n+1}(x)-\beta^n\alpha(x)\right)+\beta^m(x)\left(\alpha^{n+1}(x)-\alpha^n(x)\beta(x)\right)\right)A(x)B(x)\right]}{1+8x}\\
&= \frac{\left[\left(\alpha^m(x)\beta^n(x)\left(\beta(x)-\alpha(x)\right)+\beta^m(x)\alpha^n\left(\alpha(x)-\beta(x)\right)\right)A(x)B(x)\right]}{1+8x}\\
&= \frac{\left[(\alpha-\beta)(\beta^m\alpha^n-\alpha^m\beta^n)\right]A(x)B(x)}{1+8x}.
\end{aligned}$$

So, the proof is completed. ☐

Theorem 2.13. (Honsberger Identity) For $n, m, r \geq 0$, the Honsberger identity is given by

$$\begin{aligned}
&JQ_n JQ_m - JQ_{n+1}JQ_{m+1}\\
&= \frac{1}{1+8x}\begin{cases}\alpha^{n+m}(x)(\alpha^2(x)+1)A^2(x)+(\alpha^n\beta^m+\alpha^m\beta^n)(1+\alpha\beta)A(x)B(x)\\ +(\beta^{n+m}(\beta^2+1))B^2(x)\end{cases}.
\end{aligned}$$

Proof. The proof is easily obtained similar way to Theorem 2.12. ☐

Theorem 2.14. The exponential functions for the Jacobsthal quaternion polynomials are given by

$$F(t) = \frac{A(x)e^{\alpha(x)t} + B(x)e^{\beta(x)t}}{\alpha(x)-\beta(x)} \tag{2.15}$$

where $\alpha(x) = \frac{1-\sqrt{1+8x}}{2}$, $\beta(x) = \frac{1+\sqrt{1+8x}}{2}$, $A(x) = \frac{(\alpha(x)-1)JQ_0(x)-JQ_1(x)}{\alpha(x)-\beta(x)}$ and $B(x) = \frac{(1-\beta(x))JQ_0(x)-JQ_1(x)}{\alpha(x)-\beta(x)}$.

Proof. Using the Binet's formula (2.8) of the Jacobsthal quaternion polynomials, we obtain

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} JQ_n(x)\frac{t^n}{n!} \\ &= \frac{1}{\alpha(x)-\beta(x)}\left[\sum_{n=0}^{\infty}(\alpha^n A(x)+\beta^n B(x))\frac{t^n}{n!}\right] \\ &= \frac{A(x)}{\alpha(x)-\beta(x)}\sum_{n=0}^{\infty}\frac{\alpha t^n}{n!}+\frac{B(x)}{\alpha(x)-\beta(x)}\sum_{n=0}^{\infty}\frac{\beta t^n}{n!} \\ &= \frac{A(x)}{\alpha(x)-\beta(x)}e^{\alpha(x)t}+\frac{B(x)}{\alpha(x)-\beta(x)}e^{\beta(x)t} \\ &= \frac{A(x)e^{\alpha(x)t}+B(x)e^{\beta(x)t}}{\alpha(x)-\beta(x)}. \end{aligned}$$

So, the proof is completed. □

Theorem 2.16. The Poisson generating function for the Jacobsthal quaternion polynomials is given by

$$\mathcal{F}(t) = \frac{A(x)e^{\alpha(x)t}+B(x)e^{\beta(x)t}}{e^t(\alpha(x)-\beta(x))} \tag{2.17}$$

where $\alpha(x) = \frac{1-\sqrt{1+8x}}{2}$, $\beta(x) = \frac{1+\sqrt{1+8x}}{2}$, $A(x) = \frac{(\alpha(x)-1)JQ_0(x)-JQ_1(x)}{\alpha(x)-\beta(x)}$ and $B(x) = \frac{(1-\beta(x))JQ_0(x)-JQ_1(x)}{\alpha(x)-\beta(x)}$.

Proof. Since $\mathcal{F}(t) = e^{-t}F(t)$, the proof is obvious. □

Theorem 2.18. For $n \geq 2$, we have

$$\sum_{k=1}^{n} JQ_k(x) = \frac{1}{2x}(JQ_{n+2}(x) - JQ_2(x)). \tag{2.19}$$

Proof. By using eq. (2.3), we can obtain the recursive relation

$$JQ_{n-2}(x) = \frac{1}{2x}(JQ_n(x) - JQ_{n-1}(x)).$$

Keeping this equality in mind and by exploiting the telescoping sum, we get

$$\begin{aligned}\sum_{k=1}^{n} JQ_k(x) &= \frac{1}{2x}\sum_{k=3}^{n+2} JQ_k(x) - \frac{1}{2x}\sum_{k=2}^{n+1} JQ_k(x) \\ &= \frac{1}{2x}(JQ_{n+2}(x) - JQ_2(x)).\end{aligned}$$

So, this completes the proof of the eq. (2.19). □

Remark 2.20. If we set $x = 1$ in eq. (2.19), then we have

$$\sum_{k=1}^{n} JQ_k = \frac{1}{2}\left(JQ_{n+2} - JQ_2\right)$$

where JQ_2 is $2nd$ Jacobsthal quaternion number, i.e.,

$$\begin{aligned}JQ_2 &= 1 + (1+2x)i + (1+4x)j + (1+6x+4x^2)k \\ &= 1 + 3i + 5j + 11k.\end{aligned}$$

Theorem 2.21. For $n \geq 2$, we have

(i) $\sum_{k=1}^{n} JQ_{2k}(x) = \frac{A(x)}{\alpha(x)-\beta(x)}\left(\frac{\alpha^2(x)(\alpha^{2n}(x)-1)}{\alpha^2(x)-1}\right) + \frac{B(x)}{\alpha(x)-\beta(x)}\left(\frac{\beta^2(x)(\beta^{2n}(x)-1)}{\beta^2(x)-1}\right).$

(ii) $\sum_{k=1}^{n} JQ_{2k-1}(x) = \frac{A(x)}{\alpha(x)-\beta(x)}\left(\frac{\alpha^3(x)(\alpha^{2n}(x)-1)}{\alpha^2(x)-1}\right) + \frac{B(x)}{\alpha(x)-\beta(x)}\left(\frac{\beta^3(x)(\beta^{2n}(x)-1)}{\beta^2(x)-1}\right).$

Proof. (i) Exploiting Binet formula for the Jacobsthal quaternion polynomials, we obtain

$$\begin{aligned}\sum_{k=1}^{n} JQ_{2k}(x) &= \sum_{k=1}^{n} \frac{\alpha^{2k}(x)A(x) + \beta^{2k}(x)B(x)}{\alpha(x) - \beta(x)} \\ &= \frac{A(x)}{\alpha(x) - \beta(x)}\sum_{k=1}^{n}\alpha^{2k} + \frac{B(x)}{\alpha(x) - \beta(x)}\sum_{k=1}^{n}\beta^{2k} \\ &= \frac{A(x)}{\alpha(x) - \beta(x)}\left(\frac{\alpha^2(x)(\alpha^{2n}(x) - 1)}{\alpha^2(x) - 1}\right) + \frac{B(x)}{\alpha(x) - \beta(x)}\left(\frac{\beta^2(x)(\beta^{2n}(x) - 1)}{\beta^2(x) - 1}\right)\end{aligned}$$

□

Therefore, the proof is completed.

(ii) The proof of (ii) can be done a similar way to the proof of (i).

Prime numbers are used in many fields and the sum of consecutive primes gives interesting results. As it is known, some prime numbers are $4k+1$ or $4k+3$, where k is any integer. In addition to the sum formulas given in Theorem 2.21, let us give the sums of consecutive Jacobsthal quaternion polynomials with indices like this.

Theorem 2.22. For $n \geq 2$, we have

(i) $\sum_{k=1}^{n} JQ_{4k+1}(x) = \frac{A(x)}{\alpha(x)-\beta(x)}\left(\frac{\alpha^5(x)(\alpha^{4n}(x)-1)}{\alpha^4(x)-1}\right)+\frac{B(x)}{\alpha(x)-\beta(x)}\left(\frac{\beta^5(x)(\beta^{4n}(x)-1)}{\beta^4(x)-1}\right).$

(ii) $\sum_{k=1}^{n} JQ_{4k+3}(x) = \frac{A(x)}{\alpha(x)-\beta(x)}\left(\frac{\alpha^7(x)(\alpha^{4n}(x)-1)}{\alpha^2(x)-1}\right)+\frac{B(x)}{\alpha(x)-\beta(x)}\left(\frac{\beta^7(x)(\beta^{4n}(x)-1)}{\beta^2(x)-1}\right).$

Proof. The proofs of these equations are similar to each other. The value of the Jacobsthal quaternion polynomials in the sum expression is written using the Binet formula. If the recurrence relation is used here and the sum value of the geometric series is substituted, the desired results are obtained. □

3 Concluding Remarks

In this paper, we study the Jacobsthal quaternion polynomials. We give some results including recurrence relations, Binet's formulas, generating functions, Catalan's Identity, Cassini's Identity and so on. In addition, we present some summation formulas for these quaternion polynomials. It must be noted that for $x = 1$, the results for the Jacobsthal quaternion polynomials given in this study correspond to the Jacobsthal quaternion [19].

References

[1] Horadam, A. F. (1996) Jacobsthal Representation Numbers, The Fibonacci Quarterly, 34, 40-54.

[2] Horadam, A. F. (1996) Jacobsthal representation numbers, significance 2, 2-8.

[3] Horadam, A. F. (1997) Jacobsthal Representation Polynomials, The Fibonacci Quarterly, 35(2), 137–148.

[4] Horadam, A. F. (2002) Convolutions for Jacobsthal-Type Polynomials, The Fibonacci Quarterly, 40(3), 212–222.

[5] Catarino, P., Morgado, M. L. (2016) On generalized Jacobsthal and Jacobsthal- Lucas polynomials, An. St Univ. Ovidius Constanta, 24(3), 61–78.

[6] Djordjevic, G. B. (2000). Generalized Jacobsthal polynomials, The Fibonacci Quarterly, 38(3), 239-243.

[7] Hoggatt, V. E. Jr., Bicknell-Johnson, M. (1978) Convolution Arrays for Jacobsthal and Fibonacci Polynomials, The Fibonacci Quarterly, 16(5) 385-402.

[8] Kilic, N. (2019) On split k- Jacobsthal and k- Jacobsthal Lucas quaternions, Ars Combinatoria, 142, 129-139.

[9] Kilic, N. (2019) On k-Jacobsthal and k-Jacobsthal-Lucas Octonions, JP Journal of Algebra Number Theory and Applications, 41(1), 1–17.

[10] Tereszkiewicz, A., Wawreniuk, I. (2015) Generalized Jacobsthal polynomials and special points for them, Applied Mathematics and Computation, 268, 806–814.

[11] Hamilton, W. R. (1853) Lectures on quaternions, Dublin: Hodges and Smith.

[12] Ward, J. P. (1997) Quaternions and Cayley numbers: algebra and applications. London: Kluwer.

[13] Horadam, A. F. (1963) Complex Fibonacci numbers and Fibonacci quaternions, American Math Montly, 70, 289-291.

[14] Halıcı, S. (2013) On complex Fibonacci quaternions, Adv. Appl. Clifford Algebras, 23, 105-112.

[15] Halıcı, S. (2012) On Fibonacci quaternions, Adv. Appl. Clifford Algebras, 22, 321-327.

[16] Iyer, M. R. (1969) A note on Fibonacci quaternions, Fibonacci Quart, 7 225-229.

[17] Polatli, E. (2016) A generalization of Fibonacci and Lucas quaternions, Adv. Appl. Clifford Algebras, 26, 719-730.

[18] Polatli, E., & Kesim, S. (2015) On quaternions with generalized Fibonacci and Lucas number components, Adv. Differ. Equ, 169, 1-8.

[19] Torunbalcı-Aydın, F., & Yüce, S. (2017). A new approach to Jacobsthal quaternions. Filomat, 31(18), 5567-5579.

5. OPTIMAL LINEAR APPROXIMATION AND ISOMETRIC EXTENSIONS

Alexander Kushpel

Abstract

Let X be a Banach space with the unit ball $B(X)$ and $A \subset X$ be a convex origin-symmetric compact in X. Let $\mathrm{j} : X \to \widetilde{X}$ be an isometric extension of X. It is well-known that linear widths $\lambda_n \left(\mathrm{j}(A), \widetilde{X}\right)$ may decrease in order when compared with $\lambda_n (A, X)$ and absolute widths $\Lambda \left(A, \widehat{X}\right) = \inf_{\mathrm{j}} \left(\mathrm{j}(A), \widetilde{X}\right)$ are realized in the space $\widehat{X}$ which is the Banach space of bounded functions $f : B(X^*) \to \mathbb{R}$ on the unit ball $B(X^*)$ of the conjugate space X^*. We show that it is sufficient to use just n-dimensional extensions of X to attain absolute linear widths. This unexpected fact significantly reduces the space $\widehat{X}$. This allows us to introduce the notion of preabsolute widths. We give the respective optimal extensions explicitly and establish order estimates for preabsolute widths of a wide range of sets of smooth functions considered in [3]. In particular, in the case of super-small and super-high smoothness considered in [3] the orders of preabsolute linear widths coincide with the orders of absolute linear widths. In the intermediate cases of finite and infinite smoothness the respective orders are different.

Keywords. Optimal linear approximation, Absolute widths, Multiplier

ORCID: 0000-0002-9585-744X
e-mail: kushpel@cankaya.edu.tr

1 Introduction

Optimal linear approximation and recovery play an important role in Approximation Theory and they are connected via absolute linear widths and duality with nonlinear approximation. n-Widths were introduced in 1936 by Kolmogorov to compare the efficiency of numerical algorithms [4]. Let $(X, \|\cdot\|_X)$ be a Banach space with the unit ball $B(X)$ and $A \subset X$ be a compact, convex and origin symmetric set in X. The Kolmogorov n-width of A in X is defined as

$$d_n(A, X) = \inf_{L_n \subset X} \sup_{x \in A} \inf_{y \in L_n} \|x - y\|_X$$

Let

$$d^n(A, X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|_X ,$$

be the Gelfand n-width [2]. Here L^n runs over all subspaces of codimension at most n. We shall concentrate here on linear widths introduced in [11]. The linear n-width of A in X is defined by

$$\lambda_n(A, X) = \inf_{\mathrm{P}_n} \sup_{x \in A} \|x - \mathrm{P}_n x\|_X ,$$

where $\mathrm{P}_n : X \to X$ varies over all linear operators of rank at most n. Let X and Z be Banach spaces, $u : X \to Z$, $u \in \mathcal{L}(X, Z)$ be a bounded linear operator and u^* be its adjoint. It is well-known if u is compact or Z is reflexive (see e.g. [10], [9]) then

$$d^n(u^*) = d_n(u) \tag{1.1}$$

and

$$\lambda_n(u) = \lambda_n(u^*) ,$$

where

$$d_n(u) = d_n(uB(X), Z) = \inf_{L_n \subset X} \sup_{x \in B(X)} \inf_{y \in L_n} \|ux - y\|_X ,$$

$$\lambda_n(u) = \lambda_n(uB(X), Z) = \inf_{\mathrm{P}_n} \sup_{x \in B(X)} \|ux - \mathrm{P}_n ux\|_X .$$

Let $\left(\widetilde{X}, \mathrm{j}\right)$ be an extension of $X \subset \widetilde{X}$, where $\mathrm{j} : X \to \widetilde{X}$ is a linear isometry. It was noticed by Kolmogorov and demonstrated on a concrete example by Tikhomirov [11] that the linear n-width of A in X may decrease in an isometric extension $\widetilde{X}$ of $X \subset \widetilde{X}$, since $\widetilde{X}$ contains more subspaces to approximate A. Hence it is natural to consider

$$\Lambda_n(A, X) = \inf \lambda_n \left(\mathrm{j}(A), \widetilde{X} \right),$$

where inf is taken over all isometric extensions $\mathrm{j} : X \to \widetilde{X}$. The width $\Lambda_n(A, X)$ is the absolute linear n-width introduced by Ismagilov [2]. It is known that $\Lambda_n(A, X) = d^n(A, X)$. Moreover, the absolute linear width is realized in the so-called universal isometric extension $\widehat{X}$ which can be constructed as following. Let $B(X^*)$ be the unit ball in the dual of X and $\widehat{X}$ be the Banach space of bounded functions $f : B(X^*) \to \mathbb{R}$ with the usual norm

$$\|f(\phi)\|_{\widehat{X}} = \sup_{\phi \in B(X^*)} |f(\phi)|.$$

Clearly, $f(\phi) = \langle x, \phi \rangle \in \widehat{X}$ for any $x \in X$. By this way we get the linear isometric extension $\mathrm{j} : X \to \widehat{X}$. Observe that Gelfand n-widths are closely connected to the linear cowidths. Let $\mathbb{R}^n$ be the coding set, i.e. the set which contains information on the elements of A and $\mathcal{L}(\mathrm{lin}(A), \mathbb{R}^n)$ be a family of coding operators, $\phi : A \to \mathbb{R}^n$. Let $D \subset X$,

$$\mathrm{diam}(D, X) = \sup \{ \|x - y\|_X \, | x, y \in D \}$$

and

$$\phi^{-1}(z) = \{ y \, | y \in X, \phi(y) = z \}$$

be the diameter of D in X and preimage of $z \in X$ respectively. The linear cowidth is defined as

$$\lambda^n(A, X) = \inf_{\phi \in \mathcal{L}(\mathrm{lin}(A), \mathbb{R}^n)} \sup_{x \in A} \mathrm{diam} \left\{ \phi^{-1}(\phi(x)) \right\}.$$

Clearly,

$$\lambda^n(A, X) = 2d^n(A, X).$$

In Section 2 we demonstrate an unexpected phenomenon. Namely, we show that instead of a considerably big extension $\widehat{X}$ of X it is sufficient to use n-dimensional extensions constructed in Theorem 2.1 to attain absolute n-widths $\Lambda_n(A, X)$. Hence it is natural to introduce a new notion of pre-absolute n-widths, $\Lambda_{n,m}(A, X)$ (see Definition 2.5), where we allow m-dimensional isometric extensions of X, $0 \le m \le n$. In Section 3 we present two-side estimates for preabsolute n-widths $\Lambda_{n,m}(A, X)$ on a wide range of sets of smooth functions considered in [3]. More precisely, denote by $\mathcal{T}_n$, $n \in \mathbb{N}$ the sequence of subspaces of trigonometric polynomials with the usual order, i.e. $\mathcal{T}_n = \text{lin}\{1, \cos kx, \sin kx, k \in \mathbb{N}\}$. Consider usual spaces L_p, $1 \leq p \leq \infty$, of p-integrable functions ϕ on the unit circle $\mathbb{T}$ with the Lebesgue measure dx,

$$\|\phi\|_p = \left(\int_{\mathbb{T}} |\phi|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Let $\phi \in L_p$ with the formal Fourier series

$$\phi \sim \sum_{k=1}^{\infty} a_k(\phi) \cos kx + b_k(\phi) \sin kx,$$

where

$$a_k(\phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \cos kt dt,$$

$$b_k(\phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \sin kt dt$$

and

$$S_n(\phi, x) = \sum_{k=1}^{n} a_k(\phi) \cos kx + b_k(\phi) \sin kx \tag{1.2}$$

be its n^{th} Fourier sum. We introduce sets of smooth functions using multipliers $\Lambda = \{\lambda(k), k \in \mathbb{N}\}$ [3]. We say that $f \in \Lambda_\beta U_p$ if

$$f \sim \sum_{k=1}^{\infty} \lambda(k) \left(a_k(\phi) \cos\left(kx - \frac{\beta\pi}{2}\right) + b_k(\phi) \sin\left(kx - \frac{\beta\pi}{2}\right) \right), \tag{1.3}$$

where $\phi \in U_p = \left\{ \phi \,\middle|\, \|\phi\|_p \leq 1 \right\}$ is the unit ball in L_p. If $\beta = 0$ then we write $\Lambda_\beta = \Lambda$. If there exists $K \in L_1$ such that

$$K \sim \sum_{k=1}^{\infty} \lambda(k) \cos\left(kx - \frac{\beta\pi}{2}\right)$$

then $\Lambda_\beta U_p$ is the set of functions f representable in the form

$$f(x) = \int_{\mathbb{T}} K(x-y) \phi(y) \, dy,$$

i.e. in this case $\Lambda_\beta U_p = K * U_p$. Observe that the smoothness of the classes $\Lambda_\beta U_p$ is determined by the rate of decay of the sequence Λ. In particular, if $\lambda(k) = k^{-r}$, $\beta = r$, $r > 0$ we get standard Sobolev classes W_p^r. If $\lambda(k) =$

$\exp(-\mu k^{\gamma})$, $\beta \in \mathbb{R}$, $\mu > 0$, $0 < \gamma < 1$, then the set $\Lambda_{\beta}U_p$ consists of infinitely differentiable functions. In the case $\gamma = 1$ we get classes of analytic functions. If $\gamma > 1$, then we obtain classes of entire functions. To simplify technical notations we present our results just in the case $1 < p, q < \infty$ and $\beta = 0$.

We show here that isometric extensions may decrease the order of linear widths of ΛU_p in L_p if $1 < p < q \leq 2$ in the case of small and finite smoothness, i.e. if

$$\lambda(k) = k^{-r}, r > \left(\frac{1}{p} - \frac{1}{q}\right)_+,$$

where $(a)_+ = \max\{a, 0\}$, $a \in \mathbb{R}$. From the other side, it is shown that in the case of super-small smoothness, i.e. if

$$\lambda(k) = \phi(k)\, k^{-\left(\frac{1}{p} - \frac{1}{q}\right)_+},$$

where $\phi(k)$ is a decreasing function, $\lim_{k\to\infty} \phi(k) = 0$ and $\phi(k^s) \asymp \phi(k)$ for any fixed $s \in \mathbb{N}$, isometric extensions can not decrease the order of preabsolute linear widths $\Lambda_{n,m}(\Lambda U_p, L_q)$, $0 \leq m \leq n$. A typical example of the sequence $\lambda(k)$ is given by

$$\lambda(k) = (\ln(k+1))^{-\varrho}\, k^{-\left(\frac{1}{p} - \frac{1}{q}\right)_+}, \varrho > 0, k \in \mathbb{N}.$$

Similarly, in the case of super-high smoothness, i.e. if

$$\lambda(k) = \exp(-\mu n^{\gamma}), \mu > 0, \gamma \geq 1$$

the order of preabsolute linear widths $\Lambda_{n,m}(\Lambda U_p, L_q)$, $(p, q) \in I$, $0 \leq m \leq n$ remains the same as $\lambda_n(\Lambda U_p, L_q)$ (see (3.5)). In this sense, the results presented here complement the results obtained in [3].

For easy of notation we will put $a_n \gg b_n$ for two sequences, if $a_n > Cb_n$ for some $C > 0$ and any $n \in \mathbb{N}$ and $a_n \asymp b_n$ if $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$ and some constants $C_1 > 0$ and $C_2 > 0$.

2 Preabsolute linear widths

Our main result significantly reduces the space $\widehat{X}$ and gives an explicit representation of the extension j which is important for applications. Consider Banach space $\mathrm{lin}(A)$ with the unit ball A and $(\mathrm{lin}(A))^*$ its conjugate with the usual norm $\|\cdot\|_{(\mathrm{lin}(A))^*}$.

Theorem 2.1. Let $A \subset X$ be a convex origin symmetric compact, diam $(A, X) < \infty$, $\phi_k \in X^*$ be such that

$$\sup \{ \|x\|_X \, |x \in A, \, \langle x, \phi_k \rangle = 0, 1 \le k \le n \} \le d^n (A, X) + \epsilon, \forall \epsilon > 0$$

and $c_k : B(X^*) \to \mathbb{R}$, $1 \le k \le n$ be the functionals of the best approximation of $\phi \in B(X^*)$ by lin $\{\phi_k, 1 \le k \le n\}$ in $\|\cdot\|_{(\text{lin}(A))^*}$. Let

$$[c_k] = \begin{cases} 0, & c_k \in X^{**}, \\ c_k, & c_k \notin X^{**}. \end{cases}$$

Then

$$\lambda_n \left(\mathrm{j}(A), \overline{X} \right) = \Lambda_n (A, X),$$

where $\mathrm{j} : X \to \overline{X} = \text{lin}\{X^{**}, [c_1], \cdots, [c_n]\} \subset \widehat{X}$.

Proof. By the definition of linear width, for any extension $\mathrm{j} : X \to \widetilde{X}$ and $\epsilon > 0$ there exist $\phi_k \in \widetilde{X}^*$and $x_k \in \widetilde{X}$, $1 \le k \le n$ such that

$$\sup_{x \in A} \left\| \mathrm{j}(x) - \sum_{k=1}^{n} \langle \mathrm{j}(x), \phi_k \rangle x_k \right\|_{\widetilde{X}} \le \lambda_n \left(\mathrm{j}(A), \widetilde{X} \right) + \epsilon.$$

Consequently, by the definition of Gelfand widths (see [2]),

$$d^n (A, X) \le \sup \left\{ \|\mathrm{j}(x)\|_{\widetilde{X}} \, |x \in A, \, \langle \mathrm{j}(x), \phi_k \rangle = 0, 1 \le k \le n \right\}$$
$$\le \lambda_n \left(\mathrm{j}(A), \widetilde{X} \right) + \epsilon. \tag{2.2}$$

Also, by the definition of Gelfand width there are such $\phi_k \in X^*, 1 \le k \le n$ that

$$\sup \{ \|x\|_X \, |x \in A, \, \langle x, \phi_k \rangle = 0, 1 \le k \le n \} \le d^n (A, X) + \epsilon \tag{2.3}$$

for any $\epsilon > 0$. Let lin (A) be the Banach space with the unit ball A and $(\text{lin}(A))^*$ be its conjugate with the usual norm

$$\|\phi\|_{(\text{lin}(A))^*} = \sup \{ |\langle x, \phi \rangle| \, |x \in A \ \}.$$

Then, $\forall \phi \in B(X^*)$, by duality and (2.3) we get

$$\inf \left\{ \left\| \phi - \sum_{k=1}^{n} c_k(\phi) \phi_k \right\|_{(\text{lin}(A))^*} |c_k, 1 \le k \le n \right\}$$
$$\le \sup \{ |\langle x, \phi \rangle| \, |x \in A, \, \langle x, \phi_k \rangle = 0, 1 \le k \le n \ \}$$

$$\le d^n(A,X)+\epsilon.$$

Since $\operatorname{diam}(A,X)<\infty$ then there are such bounded functions $\phi\mapsto c_k(\phi)$, $1\le k\le n$, $\phi\in B(X^*)$ that

$$\sup\left\{\left\|\phi-\sum_{k=1}^{n}c_k(\phi)\,\phi_k\right\|_{(\operatorname{lin}(A))^*}|\phi\in B(X^*)\right\}\le d^n(A,X)+\epsilon,$$

or

$$\begin{aligned}&\sup_{\phi\in B(X^*)}\sup_{x\in A}\left|\langle x,\phi\rangle-\sum_{k=1}^{n}c_k(\phi)\langle x,\phi_k\rangle\right|\\&=\sup_{x\in A}\sup_{\phi\in B(X^*)}\left|\langle x,\phi\rangle-\sum_{k=1}^{n}c_k(\phi)\langle x,\phi_k\rangle\right|\\&=\sup_{x\in A}\left\|\mathrm{j}(x)-\sum_{k=1}^{n}\langle x,\phi_k\rangle c_k\right\|_{\widehat{X}}\le d^n(A,X)+\epsilon.\end{aligned}\tag{2.4}$$

Since $\phi\in B(X^*)$ then $\langle x,\phi\rangle\in X^{**}$ and $c_k\in\widehat{X}$. Consequently,

$$\begin{aligned}\langle x,\phi\rangle-\sum_{k=1}^{n}\langle x,\phi_k\rangle c_k&\in\operatorname{lin}\{X^{**},c_k,1\le k\le n\}\\&=\operatorname{lin}\{X^{**},\ [c_k],1\le k\le n\},\end{aligned}$$

where c_k, $1\le k\le n$ are defined by (2.4). Observe that if among c_k, $1\le k\le n$ there are linear functionals c_s on X^* then $c_s\in X^{**}$, or $[c_s]=0$. Comparing (2.2) and (2.4) we get the proof. □

Theorem 2.1 allows us to introduce the following notion.

Definition 2.5. Let $(X,\|\cdot\|_X)$ be a Banach space and $A\subset X$ be a compact, convex and origin symmetric set in X. The m-preabsolute linear n-width of A in X is defined by

$$\Lambda_{n,m}(A,X)=\inf\lambda_n\left(\mathrm{j}_m(A),\widetilde{X}\right),0\le m\le n,$$

where inf is taken over all isometric extensions

$$\mathrm{j}_m:X\to\widetilde{X}=\operatorname{lin}\{X,c_k,k\le m,c_0=0\}$$

and

$$c_k:B(X^*)\to\mathbb{R},k\le m.$$

Observe that, by Theorem 2.1,

$$\begin{aligned}\lambda_n(A,X)=\Lambda_{n,0}(A,X)\ge\Lambda_{n,1}(A,X)\ge\cdots\ge\Lambda_{n,n}(A,X)\\=\Lambda_n(A,X)=d^n(A,X).\end{aligned}\tag{2.6}$$

3 Examples and application

In this section we consider several motivating examples in the case $1 < p < q \leq 2$ to underline the dependence of preabsolute widths on smoothness. In particular, it is shown that in the cases of super-small and super-high smoothness isometric extensions can not decrease the order of linear widths.

Theorem 3.1. 1. Let $1 < p < q \leq 2$ and

$$\lambda(k) = \varphi(k)\, k^{-\left(\frac{1}{p}-\frac{1}{q}\right)_+}, k \in \mathbb{N}$$

in (1.3), where $(a)_+ = \max\{a, 0\}$, $\varphi(k)$ is a decreasing function, $\lim_{k\to\infty} \varphi(k) = 0$ and $\varphi(k^s) \asymp \varphi(k)$ for any fixed $s > 0$ (i.e. the case of super-small smoothness). Then

$$\Lambda_{n,m}(\Lambda U_p, L_q) \asymp \varphi(n), 1 < p, q < \infty, 0 \leq m \leq n. \tag{3.2}$$

2. If $\lambda(k) = k^{-r}$ where

$$\frac{1}{p} - \frac{1}{q} < r < \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{p} - \frac{1}{2}\right), 1 < p < q \leq 2$$

(i.e. the case of small smoothness) then

$$n^{\frac{p}{2(p-1)}\left(-r+\frac{1}{p}-\frac{1}{q}\right)} \ll \Lambda_{n,m}\left(W_p^r, L_q\right) \ll n^{-r+\frac{1}{p}-\frac{1}{q}}, 0 \leq m \leq n. \tag{3.3}$$

3. If $\lambda(k) = k^{-r}$ where

$$r > \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{p} - \frac{1}{2}\right)$$

(i.e. the case of finite smoothness) then

$$n^{-r} \ll \Lambda_{n,m}\left(W_p^r, L_q\right) \ll n^{-r+\frac{1}{p}-\frac{1}{q}}, 0 \leq m \leq n. \tag{3.4}$$

4. Let $\lambda(k) = \exp(\mu k^\gamma)$, $\mu > 0, 0 < \gamma < 1$ in (1.3) (i.e. the case of infinite smoothness) then

$$\exp(-\mu n^\gamma) \ll \Lambda_{n,m}(\Lambda U_p, L_q) \ll \exp(-\mu n^\gamma)\, n^{(1-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}, 0 \leq m \leq n. \tag{3.5}$$

5. Let $\lambda(k) = \exp(\mu k^\gamma)$, $\mu > 0, \gamma \geq 1$ in (1.3) (i.e. the case of super-high smoothness) then

$$\Lambda_{n,m}(\Lambda U_p, L_q) \asymp \exp(-\mu n^\gamma), \mu > 0, \gamma \geq 1, 0 \leq m \leq n. \tag{3.6}$$

Proof. Let us consider the case of super-small smoothness (3.2). It was shown in [3] that in this case

$$d^n\left(\Lambda U_p, L_q\right) \asymp \lambda_n\left(\Lambda U_p, L_q\right)$$

$$\asymp \sup_{f\in\Lambda U_p}\left\|f - S_n\left(f\right)\right\|_q \asymp \varphi\left(n\right).$$

Consequently, in this case

$$\Lambda_{n,m}\left(\Lambda U_p, L_q\right) \asymp \varphi\left(n\right), 1 < p, q < \infty \tag{3.7}$$

for any $0 \le m \le n$.

Let $\Lambda U_p = W_p^r$ be Sobolev class. In this case $\lambda\left(k\right) = k^{-r}$. We show (3.4). It is known [3] that

$$\lambda_n\left(W_p^r, L_q\right) \ll \sup_{f\in W_p^r}\left\|f - S_n\left(f\right)\right\|_q \ll n^{-r+\frac{1}{p}-\frac{1}{q}}, 1 < p < q \le 2, r > \frac{1}{p}-\frac{1}{q}, \tag{3.8}$$

where S_n is defined by (1.2). By the Theorem 3.1, (2.6) and (1.1) for any $0 \le m \le n$ we have

$$\Lambda_{n,m}\left(W_p^r, L_q\right) \ge \Lambda_{n,n}\left(W_p^r, L_q\right) = \Lambda_n\left(W_p^r, L_q\right)$$

$$= d^n\left(W_p^r, L_q\right) = d_n\left(W_{q'}^r, L_{p'}\right),$$

where

$$p^{'} = \begin{cases} \frac{p}{p-1}, & 1 < p < \infty, \\ 1, & p = \infty, \\ \infty, & p = 1. \end{cases}$$

Since $1 < p < q \le 2$ then $2 \le q^{'} < p^{'} < \infty$ and if

$$r > \frac{1}{2}\left(\frac{1}{q^{'}} - \frac{1}{p^{'}}\right) / \left(\frac{1}{2} - \frac{1}{p^{'}}\right) = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{p} - \frac{1}{2}\right) \tag{3.9}$$

then

$$d_n\left(W_{q'}^r, L_{p'}\right) \asymp n^{-r}.$$

Clearly,

$$\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{p} - \frac{1}{2}\right) > \frac{1}{p} - \frac{1}{q}.$$

Hence

$$n^{-r} \ll \Lambda_{n,m}\left(W_p^r, L_q\right) \ll n^{-r+\frac{1}{p}-\frac{1}{q}}, 0 \le m \le n$$

if (3.9) is satisfied. This proves (3.4).

To show (3.3) we remark that if $2 \le p < q < \infty$ then [1], [12], [3]

$$d_n\left(W_p^r, L_q\right) \asymp n^{\frac{q}{2}\left(-r+\frac{1}{p}-\frac{1}{q}\right)},$$

where

$$\frac{1}{p}-\frac{1}{q} < r < \frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right).$$

Consequently, by (2.6) and (1.1) we get

$$\Lambda_{n,m}\left(W_p^r, L_q\right) \ge d^n\left(W_p^r, L_q\right) = d_n\left(W_{q'}^r, L_{p'}\right)$$

$$\asymp n^{\frac{p'}{2}\left(-r+\frac{1}{q'}-\frac{1}{p'}\right)} = n^{\frac{p}{2(p-1)}\left(-r+\frac{1}{p}-\frac{1}{q}\right)}.$$

The respective upper bounds follow from (3.8). This proves (3.3).

The case (3.5) can be treated similarly. Namely, since

$$\lambda_{2n}\left(\Lambda U_p, L_q\right) \ll \sup_{f\in W_p^r}\|f-S_n(f)\|_q \ll \exp\left(-\mu n^\gamma\right) n^{(1-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)_+},$$

where

$$1 < p < q \le 2, \mu > 0, 0 < \gamma < 1.$$

[3], [5], [6], [7], [8] and by (1.1)

$$d^{2n}\left(\Lambda U_p, L_q\right) \asymp d_{2n}\left(\Lambda U_{p'}, L_{q'}\right) \asymp \exp\left(-\mu n^\gamma\right),$$

$$2 \le p' < q' < \infty$$

then

$$\exp\left(-\mu n^\gamma\right) \ll \Lambda_{2n,2m}\left(\Lambda U_p, L_q\right) \ll \exp\left(-\mu n^\gamma\right) n^{(1-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}, 0 \le m \le n.$$

Finally, consider the case of super-high smoothness (3.6). Namely, if

$$\lambda(k) = \exp\left(\mu k^\gamma\right), \mu > 0, \gamma \ge 1.$$

In this case [5]

$$\lambda_{2n}\left(\Lambda U_p, L_q\right) \ll \sup_{f\in W_p^r}\|f-S_n(f)\|_q \ll \exp\left(-\mu n^\gamma\right), 1 < p, q < \infty$$

and

$$d^{2n}\left(\Lambda U_p, L_q\right) \asymp d_{2n}\left(\Lambda U_{p'}, L_{q'}\right) \asymp \exp\left(-\mu n^\gamma\right), 1 < p, q < \infty.$$

Consequently,

$$\Lambda_{n,m}\left(\Lambda U_p, L_q\right) \asymp \exp\left(-\mu n^\gamma\right), \mu > 0, \gamma \ge 1$$

for any $0 \le m \le n$. □

Acknowledgements

The Author wishes to thank the organisers of the $2^{\underline{nd}}$ Gaussian Symposium, Munich, 1993, where Theorem 1 was reported by the author. Also, I wish to thank referees and Communicating Editor for valuable comments.

References

[1] Galeev, È. F. M. (2011). Kolmogorov and linear diameters of functional classes and finite dimensional sets. Vladikavkazskii Matematicheskii Zhurnal, 13(2), 3-14.

[2] Ismagilov, R. S. M. (1974). Diameters of sets in normed linear spaces and the approximation of functions by trigonometric polynomials. Uspekhi Matematicheskikh Nauk, 29(3), 161-178.

[3] Jarad, F., Kushpel, A., & Taş, K. (2020). On the optimality of the trigonometric system. Journal of Complexity, 56, 101429.

[4] Kolmogorov, A.N. (1936). Über die beste annaherung von funktionen einer gegebenenfunklasse. Annals of Mathematics 37, 1, 107-110.

[5] Kushpel, A. (1989). Estimates of widths of classes of analytic functions, Ukrainian Math. J. 41 (4) 493-496.

[6] Kushpel, A. (1990). Estimation of the widths of classes of smooth functionsin the space L_q, Ukrainian Math. J. 42 (2) 248-249.

[7] Kushpel, A. Estimates of the Bernstein widths and their analogs, Ukrainian Math. J. 45(1) (1993) 59-65.

[8] Kushpel, A., & Levesley, J., & Wilderotter, K. (1998). On the asymptotically optimal rate of approximation of multiplier operators, Constr. Approx. 14 169-185.

[9] Kushpel, A. (2023). Optimal recovery and volume estimates. Journal of Complexity 79 101780.

[10] Lorentz, G. G., von Golitschek, M., & Makovoz, Y. (1996). Constructive approximation: advanced problems (Vol. 304). Berlin: Springer.

[11] Tikhomirov, V.M. (1960). Diameters of sets in function spases and the theory of best approximation. Russian Math. Surveys 15, 75-111.

[12] Vybiral, J. (2008). Widths of embeddings in function spaces. Journal of Complexity 24, 545-570.

6. APPLICATION OF sk-SPLINES IN THE OPTIMAL RECONSTRUCTION OF SMOOTH FUNCTIONS

Alexander Kushpel **Nurgül Gökgöz**
Erkan Murat Türkan

Abstract

We consider the problem of optimal approximation of sets of smooth functions using linear methods. It is shown that sk-splines with uniform nodes and points of interpolation realise the best order of approximation on the class of linear methods of recovery using linear information.

Keywords. Reconstruction, sk-spline, Smooth function.

ORCID: 0000-0002-9585-744X
e-mail: kushpel@cankaya.edu.tr

ORCID: 0000-0002-9640-4194
e-mail: ngokgoz@cankaya.edu.tr

ORCID: 0000-0002-2923-6661
e-mail: emturkan@cankaya.edu.tr

1 Introduction

One of the central problems of Applied Mathematics is to find optimal methods of approximation of functions on a wide class of numerical algorithms. The most simple and important methods in this range of problems are linear. In particular, let us consider a set A of continuous (smooth) functions on some domain Ω. Assume that we can take values of functions $f \in A$ at some points $x_k \in \Omega$, $1 \leq k \leq n$. Next, using this information we need to approximate this function in the best possible way. To approach this important in applications problem we introduce the respective extremal problem in an abstract settings and then give an explicit solution in several important cases.

Let $(X, \|\cdot\|)$ be a Banach space with the unit ball

$$B(X) = \{x \,|x \in X, \; \|x\| \leq 1\}$$

and $X^{'}$be its dual. Let $y_k \in X^{'}$, $1 \leq k \leq n$ and T_n be an information operator,

$$\begin{array}{ccc} T_n : X & \longrightarrow & \mathbb{R}^n \\ T_n x & \mapsto & (\langle x, y_1 \rangle, \cdots, \langle x, y_n \rangle). \end{array}$$

Let S_n,

$$\begin{array}{ccc} S_n : \mathbb{R}^n & \longrightarrow & X \\ S_n \circ T_n x & \mapsto & \sum_{k=1}^{n} \langle x, y_k \rangle \, \phi_k. \end{array}$$

be a linear recovery operator, $\phi_k \in X$, $1 \leq k \leq\leq n$. Let $A \subset X$. Consider the following extremal problem,

$$E(A, X, n) := \inf_{S_n, T_n} \sup_{x \in A} \|x - S_n \circ T_n x\|.$$

Let, in particular, $X = L_p(-\pi, \pi)$ be the standard space of 2π-periodic and p-integrable functions f on $(-\pi, \pi)$ with the norm

$$\|f\|_p := \left(\int_{-\pi}^{\pi} |f|^p \, dx \right)^{\frac{1}{p}}, 1 < p < \infty$$

and A be the usual set of functions representable in the convolution form,

$$f(\cdot) = \int_{-\pi}^{\pi} K(\cdot - t)\, \varphi(t)\, dt,$$

where $\varphi \in L_p(-\pi, \pi)$ and

$$K(t) = a(0) + \sum_{k=1}^{\infty} a(k) \cos kt.$$

Let $\varphi \in L_p$ with the formal Fourier series

$$\varphi(t) \sim a_0(\varphi) + \sum_{k=1}^{\infty} a_k(\varphi) \cos kt + b_k(\varphi) \sin kt,$$

where

$$a_k(\varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\cos ktdt,\, b_k(\varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\sin ktdt,\, k \in \mathbb{N},$$

$$a_0(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)\, dt$$

and

$$S_n(\varphi, t) = a_0(\varphi) + \sum_{k=1}^{n} a_k(\varphi)\cos kt + b_k(\varphi)\sin kt$$

be the n^{th} Fourier sum. It is easy to check that

$$f = K * \varphi \sim a_0(\varphi)\, a(0) + \sum_{k=1}^{\infty} a(k)\left(a_k(\varphi)\cos kt + b_k(\varphi)\sin kt\right).$$

We say that $f \in \Lambda B(L_p(-\pi,\pi))$ if $\varphi \in B(L_p(-\pi,\pi))$. In particular, if $K \in L_1$ then

$$\Lambda B(L_p(-\pi,\pi)) = K * B(L_p(-\pi,\pi)).$$

The rate of decay of the sequence $\{a(k), k \in \mathbb{N}\}$ governs the smoothness of the function class $A = K * B(L_p(-\pi,\pi))$. In particular, if $a(k) = k^{-\gamma}, \gamma > 0$ then we get standard Sobolev classes of fractional smoothness γ. If $a(k) = \exp\left(-\alpha k^{\beta}\right)$, $\alpha > 0$, $0 < \beta < 1$ we get sets of infinitely differentiable functions. If $\beta = 1$ then A is a class of analytic functions and if $\beta > 1$ then we get sets of entire functions (see [3], [4] for more information). We show that in all these cases

$$E(A, X, n) \gg a(n),\, n \to \infty. \tag{1.1}$$

Let us turn to the upper bounds. For a given $n \in \mathbb{N}$ let

$$\Lambda_{2n} = \left\{t_k = \frac{\pi k}{n}, 1 \le k \le 2n\right\}$$

be a partition of $(0, 2\pi]$ and $K : [0, 2\pi) \to \mathbb{R}$ be a continuous 2π-periodic function. Denote by $SK(\Lambda_{2n})$ the space of sk-splines, i.e.

$$SK(\Lambda_{2n}) = \text{lin}\left\{K(t - t_k), t_k \in \Lambda_{2n}\right\}.$$

In particular, if $a(k) = k^{-\gamma}$, $\gamma = 2s$, $s \in \mathbb{N}$, i.e.

$$K(t) = D_r(t) = \sum_{k=1}^{\infty} k^{-\gamma}\cos kt$$

is the Bernoulli monospline then $SK(\Lambda_{2n})$ is the space of polynomial splines of order $2s-1$, defect 1 with knots x_k. Let

$$K(t) \sim a(0) + \sum_{k=1}^{\infty} a(k) \cos kt.$$

It is known [2] that the problem of interpolation at $t_k \in \Lambda_{2n}$ by sk-splines has a unique solution if

$$a(k) = k^{-\gamma}, \gamma > 1, a(k) = \exp\left(-\alpha k^{\beta}\right), \alpha > 0, \beta > 0 \tag{1.2}$$

and fundamental splines can be represented as

$$\widetilde{sk}(t) = \frac{1}{2n} + \frac{1}{2n} \sum_{j=1}^{2n-1} \frac{\rho_j(t)}{\rho_j(0)},$$

where

$$\rho_j(t) = \sum_{\nu=1}^{2n} \cos\left(\frac{\pi\nu j}{n}\right) K\left(t - \frac{\pi\nu}{n}\right).$$

Hence, sk-spline interpolant $sk(f,t)$ has the form

$$sk(f, \Lambda_{2n}, t) = \sum_{k=1}^{2n} f(t_k)\, \widetilde{sk}(t - t_k).$$

It is known [3], [4] that in the cases (1.2)

$$\sup_{f \in K * B(L_p(-\pi,\pi))} \|f(\cdot) - sk(f, \Lambda_{2n}, \cdot)\|_p \ll a(n), 1 < p < \infty. \tag{1.3}$$

Comparing (1.1) and (1.3) we conclude that sk-splines with knots and points of interpolation t_k, $1 \le k \le 2n$ give the best possible order of convergence on the class of all linear methods of coding and recovery, that is among all methods of approximation of the form

$$S_{2n} \circ T_{2n} f(\cdot) = \sum_{k=1}^{2n} \langle f, y_k \rangle \psi_k(\cdot),$$

where $\psi_k \in L_p(-\pi,\pi)$, $1 \le k \le 2n$ is an arbitrary system of functions and $\langle f, y_k \rangle \in (L_p(-\pi,\pi))'$, $1 \le k \le 2n$.

2 The results

Theorem 2.1. Let $A \subset X$ and dim lin $(A) \geq n+1$. Assume that for some $\rho_n > 0$ we have

$$\rho_n B\left(X\right) \cap M_{n+1} \subset A,$$

for some subspace $M_{n+1} \subset X$, dim $M_{n+1} = n+1$. Then

$$E\left(A, X, n\right) \geq \rho_n.$$

Proof. Let $\langle x, y_k \rangle \in X^{'}$, $1 \leq k \leq n$ be arbitrary functionals. Then for any T_n and S_n we get

$$\sup_{x \in A} \|x - S_n \circ T_n x\| \geq \sup\left\{\|x\| \,|x \in A \cap L^n\right\}, \tag{2.2}$$

where

$$L^n = \{x\,|x \in A,\ \langle x, y_k \rangle = 0, 1 \leq k \leq n\}$$

is a subspace of codimension $\leq n$. For a fixed $M_{n+1} \subset X$ we have

$$\dim\left(M_{n+1} \cap L^n\right) \geq 1.$$

Clearly, if $A \subseteq B \subset X$ then

$$E\left(B, X, n\right) \geq E\left(A, X, n\right)$$

and

$$E\left(aA, X, n\right) = |a|\, E\left(A, X, n\right)$$

for any $a \in \mathbb{R}$. Consequently, applying (2.2) we obtain

$$\begin{aligned} E\left(A, X, n\right) &\geq E\left(A \cap M_{n+1}, X, n\right) \\ &\geq E\left(\rho_n B\left(X\right) \cap M_{n+1}, X, n\right) \\ &= \rho_n E\left(B\left(X\right) \cap M_{n+1}, X, n\right) \\ &\geq \rho_n \sup\left\{\|x\|\,|x \in B\left(X\right) \cap \left(L^n \cap M_{n+1}\right)\right\} \\ &\geq \rho_n. \end{aligned}$$

□

To apply Theorem 2.1 we need the following statement established in [1] (see also [5]).

Lemma 2.3. Let $1 < p < \infty$, then

$$\|\Lambda \,| L_p \to L_p \| \leq \chi_p \left(\sum_{k=0}^{\infty} |\lambda(k) - \lambda(k+1)| + \sup_{m \in \mathbb{N}} |\lambda(m)| \right),$$

where

$$\chi_p = 1 + 2 \left\{ \begin{array}{ll} \cot \frac{\pi}{2\pi}, & 2 < p < \infty, \\ \tan \frac{\pi}{2\pi}, & 1 < p \leq 2. \end{array} \right\}$$

Theorem 2.4. Let $\{a(k), k \in \mathbb{N} \cup \{0\}\}$, $a(k) \neq 0$ be a decreasing sequence of real numbers for some $k \geq M$. Then

$$E(K * B(L_p(-\pi, \pi)), L_p(-\pi, \pi), 2n - 1) \geq C_p^{-1} a(n), 1 < p < \infty.$$

Proof. We reduce the problem to a finite dimensional one. Consider multiplier operator

$$\Lambda_n^{-1} = \left\{ \frac{1}{a(0)}, \frac{1}{a(1)}, \frac{1}{a(2)}, \cdots, \frac{1}{a(n)}, 0, 0, \cdots \right\}.$$

Using Lemma 2.3 we get

$$\|\Lambda \,| L_p \to L_p \| \leq C_p \frac{1}{a(n)}.$$

This implies

$$C_p^{-1} a(n) B(L_p(-\pi, \pi)) \cap \mathcal{T}_{2n+1} \subset K * B(L_p(-\pi, \pi)),$$

where

$$\mathcal{T}_{2n+1} = \text{lin}\{1, \cos kt, \sin kt, 1 \leq k \leq n\}.$$

Clearly, $\dim \mathcal{T}_{2n+1} = 2n + 1$. Applying Theorem 2.1 we get

$$E(K * B(L_p(-\pi, \pi)), L_p(-\pi, \pi), 2n - 1) \geq C_p^{-1} a(n).$$

□

Theorem 2.5. Let $\{a(k), k \in \mathbb{N}\}$ be such as in (1.2) then

$$E(K * B(L_p(-\pi, \pi)), L_p(-\pi, \pi), 2n) \asymp a(n),$$

$$n \to \infty, 1 < p < \infty.$$

Proof. By (1.3) and Theorem 2.4 we get

$$E\left(K * B\left(L_p\left(-\pi, \pi\right)\right), L_p\left(-\pi, \pi\right), 2n\right) \gg a\left(n\right)$$

and

$$E\left(K * B\left(L_p\left(-\pi, \pi\right)\right), L_p\left(-\pi, \pi\right), 2n\right) \ll a\left(n\right)$$

as $n \to \infty$. □

References

[1] Kushpel, A. (1988). On an estimate of widths of sets in some function spaces, In: Somme problems in the Theory of Approximation of functions, Kiev, Inst. Math. Acad. Nauk Ukraine. SSR, 92-95.

[2] Kushpel, A. (1989). Sharp estimates of the widths of convolution classes, Math. USSR Izvestiya, American Mathematical Society, **33**, 631-649.

[3] Kushpel, A. (2008). Convergence of sk-splines I, *Int. J. of Pure and Applied Mathematics*, **45**, 87-101.

[4] Kushpel, A. (2008). Convergence of sk-splines II, *Int. J. of Pure and Applied Mathematics*, **45**, 103-119.

[5] Jarad, F., & Kushpel, A., & Tas, K. (2020). On the optimality of the trigonometric system, Journ. of Complexity, 56, 101429.

7. CALCULATION OF COVID-19 CASES USING ADOMIAN DECOMPOSITION METHOD

Nurgül Gökgöz

Abstract

In the recent years, analysis of fractional order equations showed that they represent some different behaviors that integer order systems may not represent and therefore they are thought to be advantageous. Thus they are frequently preferred instead of the integer order equations to model real world problems. One example of the fractional order derivatives is the Caputo derivative which allows a more suitable approach to calculate if the initial conditions are given. We investigate the Caputo type fractional order system model given in [1],

$$^{C}D_t^{\alpha}[S(t)] = -\beta\frac{S(t)I(t)}{N}, \qquad ^{C}D_t^{\alpha}[I(t)] = \beta\frac{I(t)S(t)}{N} - (\gamma+\kappa)I(t)$$
$$^{C}D_t^{\alpha}[R(t)] = \gamma I(t), \qquad ^{C}D_t^{\alpha}[D(t)] = \kappa I(t)$$

where $0 < \alpha \leq 1$. In this equation, S, I, R and D corresponds to the susceptible, infected, recovered and dead number of people. Parameters β, γ ve κ represents the average number of contacts per person per time, recovery rate and death rate, respectively.
The analytical solutions of fractional order nonlinear systems, like the one given above, can not be evaluated most of the time. In such cases obtaining numerical solutions is the first way. However, the discretization causes round off errors and requires computer time and memory. Those factors cause the loss of accuracy. Therefore, we investigate the solutions of the above model in terms of Adomian decomposition method.

Keywords. Fractional order models, Caputo fractional derivative, Adomian decomposition method

ORCID: 0000-0002-9640-4194
e-mail: ngokgoz@cankaya.edu.tr

1 Introduction

Mathematical models serve a systematic way to understand nature, technology and science and they provide us insight about the evolution, existence, stability and control of the system in question. By the outbreak of

COVID-19, the importance of mathematical models have proven to be useful. Even though the pandemic is considered as almost over, many authorities from various disciplines warn public about the upcoming similar pandemics/endemics. Therefore, all the models and solution concepts studied for COVID-19 will shed a light for the next widespread disease.
A wide range of mathematical models for COVID-19 that takes into account different perspectives or mathematical approaches have been developed within this period of time. Integer order [6],[7], [8] or fractional order compartmental models [1], [9], [10]. Apart from compartmental models, also machine learning or deep learning algorithms have been widely used to predict number of cases [11], [12], [13]. Most nonlinear models do not result in an analytical solution and thus applying a numerical method to solve the COVID-19 model is the most common way [14], [1]. Apart from those numerical methods, a recently developed tool, Adomian Decomposition [15], has proven to be a very advantageous way to obtain a semi-analytical solution. There are some works in the literature that takes into account the integer order Adomian decomposition solutions of COVID-19 models[5]. In this work, we consider fractional order Adomian decomposition to obtain a solution of the SIRD model.

2 Fractional order Adomian Decomposition Method

In the Adomian decomposition case, it is assumed that [4]

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Moreover, the nonlinear term $F(u(x,t))$ is represented as

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n$$

where A_n are called as the Adomian polynomials [4]. Adomian has discovered his decomposition method for different types of nonlinearities [15], [16], [17]. The complexity in this method is to calculate the Adomian polynomials and therefore some methodologies have been developed in this direction [4], [18]. Also, fractional counterparts have been developed [1]. In this work, fractional Adomian decomposition of [1] that makes use of Caputo derivative is employed.

Definition 2.1 ([19],[20],[1]). Let $f \in C^m_{-1}$, $m \in \mathbb{N} \cup \{0\}$. Then the left-sided Caputo fractional derivative of f is defined as

$$D^\mu f(x) = \begin{cases} I^{m-\mu} f^m(x), & m-1 < \mu \leq m, \quad m \in \mathbb{N} \\ \frac{d^m f(x)}{dx^m}, & \mu = m \end{cases}$$

Moreover, it is stated that [19],[20],[1],

$$\mathcal{I}^\mu \mathcal{I}^\nu f = \mathcal{I}^{\mu+\nu} f, \quad \mu, \nu \geq 0, f \in C^\alpha, \quad \alpha \geq 1$$

$$\mathcal{I}^\mu x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\lambda+\mu+1)} x^{\gamma+\mu}, \quad \mu > 0, \quad \gamma > -1, \quad x > 0.$$

3 Solution using Fractional Adomian Decomposition Method

The model of Nisar, et al.,[1],

$$^C D_t^\alpha [S(t)] = -\beta \frac{S(t)I(t)}{N} \tag{3.1}$$

$$^C D_t^\alpha [I(t)] = \beta \frac{I(t)S(t)}{N} - (\gamma + \kappa) I(t) \tag{3.2}$$

$$^C D_t^\alpha [R(t)] = \gamma I(t) \tag{3.3}$$

$$^C D_t^\alpha [D(t)] = \kappa I(t) \tag{3.4}$$

has proven to have a unique solution in the aforementioned paper [1]. Therefore, we obtain a solution in the form

$$S(t) = S(0) + \mathcal{I}^\alpha \left(\beta \frac{S(t)I(t)}{N} \right) \tag{3.5}$$

$$I(t) = I(0) + \mathcal{I}^\alpha \left(\beta \frac{I(t)S(t)}{N} - (\gamma + \kappa) I(t) \right) \tag{3.6}$$

$$R(t) = R(0) + \mathcal{I}^\alpha \left(\gamma I(t) \right) \tag{3.7}$$

$$D(t) = D(0) + \mathcal{I}^\alpha \left(\kappa I(t) \right) \tag{3.8}$$

where $S(0)$, $I(0)$, $R(0)$ and $D(0)$ are initial conditions and $\mathcal{I}^\alpha$ corresponds to the Caputo integral. The complete solution in the series form will correspond to

$$\begin{aligned} S(t) &= S_0 + S_1 + S_2 + S_3 + \dots, \\ I(t) &= I_0 + I_1 + I_2 + I_3 + \dots, \\ R(t) &= R_0 + R_1 + R_2 + R_3 + \dots, \\ D(t) &= D_0 + D_1 + D_2 + D_3 + \dots \end{aligned}$$

In order to apply Adomian decomposition method we define the right-hand side terms as,

$$\begin{aligned}
N_1 &= \beta \frac{S(t)I(t)}{N} = \sum_{j=1}^{\infty} A_{1j}, \\
N_2 &= \beta \frac{I(t)S(t)}{N} - (\gamma + \kappa)I(t) = \sum_{j=1}^{\infty} A_{2j}, \\
N_3 &= \gamma I(t) = \sum_{j=1}^{\infty} A_{3j}, \\
N_4 &= \kappa I(t) = \sum_{j=1}^{\infty} A_{4j}.
\end{aligned}$$

Then, we calculate Adomian polynomials as

$$\begin{aligned}
A_{10} &= \frac{\beta}{N} S_0 I_0, \\
A_{11} &= \frac{\beta}{N} S_0 I_1 + \frac{\beta}{N} S_1 I_0, \\
A_{12} &= \frac{\beta}{N} S_0 I_2 + \frac{\beta}{N} S_1 I_1 + \frac{\beta}{N} S_2 I_0, \\
A_{13} &= \frac{\beta}{N} S_0 I_3 + \frac{\beta}{N} S_1 I_2 + \frac{\beta}{N} S_2 I_1 + S_3 I_0 \\
A_{14} &= \frac{\beta}{N} S_0 I_4 + \frac{\beta}{N} S_1 I_3 + \frac{\beta}{N} S_2 I_2 + \frac{\beta}{N} S_3 I_1 + \frac{\beta}{N} S_4 I_0, \\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
A_{20} &= \frac{\beta}{N} S_0 I_0 - (\gamma + \kappa) I_0, \\
A_{21} &= \frac{\beta}{N} S_0 I_1 + \frac{\beta}{N} S_1 I_0 - (\gamma + \kappa) I_1, \\
A_{22} &= \frac{\beta}{N} S_0 I_2 + \frac{\beta}{N} S_1 I_1 + \frac{\beta}{N} S_2 I_0 - (\gamma + \kappa) I_2, \\
A_{23} &= \frac{\beta}{N} S_0 I_3 + \frac{\beta}{N} S_1 I_2 + \frac{\beta}{N} S_2 I_1 + \frac{\beta}{N} S_3 I_0 - (\gamma + \kappa) I_3, \\
A_{24} &= \frac{\beta}{N} S_0 I_4 + \frac{\beta}{N} S_1 I_3 + \frac{\beta}{N} S_2 I_2 + \frac{\beta}{N} S_3 I_1 + \frac{\beta}{N} S_4 I_0 - (\gamma + \kappa) I_4, \\
&\vdots
\end{aligned}$$

and for A_{3j} and A_{4j} we obtain the following Adomian polynomials as

$$\begin{aligned}
A_{30} &= \gamma I_0 \text{ and } A_{40} = \kappa I_0,\\
A_{31} &= \gamma I_1 \text{ and } A_{41} = \kappa I_1,\\
A_{32} &= \gamma I_2 \text{ and } A_{42} = \kappa I_2,\\
A_{33} &= \gamma I_3 \text{ and } A_{43} = \kappa I_3,\\
A_{34} &= \gamma I_4 \text{ and } A_{44} = \kappa I_4,\\
&\vdots
\end{aligned}$$

One can calculate the Adomian polynomials using different programming languages like Mathematica [2] or MATLAB [3]. In paper [1], the initial values are considered to be $S_0 = N - 6$ where $N = 11000000/250$, $I_0 = 1$, $R_0 = 0$ and $D_0 = 0$ and moreover, β which is the average number of contact per person per time is taken to be 5, γ which represents the recovery rate is taken as 0.5 and κ which corresponds to the death rate is taken as 3.5. Therefore, Adomian decomposition series has the following terms

$$\begin{cases} S(0) = S_0 = N - 6 \\ S_{m+1} = \mathcal{I}^{\alpha} A_{1m} \end{cases} \qquad \begin{cases} I(0) = I_0 = 1 \\ I_{m+1} = \mathcal{I}^{\alpha} A_{2m} \end{cases}$$

$$\begin{cases} R(0) = R_0 = 0 \\ R_{m+1} = \mathcal{I}^{\alpha} A_{3m} \end{cases} \qquad \begin{cases} D(0) = D_0 = 0 \\ D_{m+1} = \mathcal{I}^{\alpha} A_{4m}, \quad m = 0, 1, 2 \end{cases}$$

In the first iteration, we obtain

$$\begin{aligned}
S_1 &= \mathcal{I}^{\alpha} A_{10} = \frac{\beta}{N} S_0 I_0 \frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
I_1 &= \mathcal{I}^{\alpha} A_{20} = \left(\frac{\beta}{N} S_0 I_0 - (\gamma+\kappa) I_0\right) \frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
R_1 &= \mathcal{I}^{\alpha} A_{30} = \gamma I_0 \frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
D_1 &= \mathcal{I}^{\alpha} A_{40} = \kappa I_0 \frac{t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

With a similar manner, in the second iteration we get,

$$\begin{aligned}
S_2 &= \mathcal{I}^{\alpha} A_{11} = \frac{\beta}{N}\mathcal{I}^{\alpha}\left(S_0 I_1 + I_0 S_1\right),\\
I_2 &= \mathcal{I}^{\alpha} A_{21} = \mathcal{I}^{\alpha}\left(\frac{\beta}{N}S_0 I_1 + \frac{\beta}{N}S_1 I_0 - (\gamma+\kappa)I_1\right),\\
R_2 &= \mathcal{I}^{\alpha} A_{31} = \mathcal{I}^{\alpha}\left(\gamma I_1\right),\\
D_2 &= \mathcal{I}^{\alpha} A_{41} = \mathcal{I}^{\alpha}\left(\kappa I_1\right).
\end{aligned}$$

Th algorithm to calculate Adomian polynomials can be found in [2] and [4]. Finally, we obtain the solutions as

$$\begin{aligned}
S(t) &= (N-6) + a_1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + a_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots,\\
I(t) &= 1 + b_1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + b_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + b_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots,\\
R(t) &= c_1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + c_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + c_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots,\\
D(t) &= d_1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + d_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + d_3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots.
\end{aligned}$$

where a_i, b_i, c_i, d_i are coefficients obtained from iterations, parameter values and first few of them are listed as follows:

$$\begin{aligned}
&a_1 = 4.9993;\;\; a_2 = 24.9933;\;\; a_3 = 104.9669,\\
&b_1 = 0.9993;\;\; b_2 = 20.9960;\;\; b_3 = 20.9828,\\
&c_1 = 0.5000;\;\; c_2 = 0.4997.9933;\;\; c_3 = 10.4980,\\
&d_1 = 3.5000;\;\; d_2 = 3.4976;\;\; d_3 = 73.4861.
\end{aligned}$$

4 Conclusion

This work aims the solution of a nonlinear system using fractional order Adomian Decomposition method which is considered as semi-analytical. Instead of using a numerical solution which may cause round-off error, etc. we provide an alternative method for simulations of the considered model. As discussed previously in this work, Adomian polynomials are hard to obtain. However, there are works in the literature to gather those polynomials by using computer programs. Therefore, application becomes a lot easier in this case. On the other hand, a detailed analysis of the Adomian method in terms of error bound will be calculated and moreover, its advantages and disadvantages with respect to other numerical methods with applications will be considered in the future.

References

[1] Nisar, K. S., & Ahmad, S., & Ullah, A., & Shah, K., & Alrabaiah, H., & Arfan, M. (2021). Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data. Results in Physics, 21, 103772.

[2] Jafari, H., & Daftardar-Gejji, V. (2006). Solving a system of nonlinear fractional differential equations using Adomian decomposition. Journal of Computational and Applied Mathematics, 196(2), 644-651.

[3] Fatoorehchi, H., & Abolghasemi, H. (2011). On calculation of Adomian polynomials by MATLAB. J. Appl. Comput. Sci. Math, 5, 85-88.

[4] Wazwaz, A. M. (2000). A new algorithm for calculating Adomian polynomials for nonlinear operators. Applied Mathematics and computation, 111(1), 33-51.

[5] Agarwal, G., & Mohan, M., & Menon, A. M., & Sharma, A., & Dakal, T. C., & Purohit, S. D. (2022). Analysis of the Adomian decomposition method to estimate the COVID-19 pandemic. In Methods of Mathematical Modeling (pp. 173-187). Academic Press.

[6] Adiga, A., & Dubhashi, D., & Lewis, B., & Marathe, M., & Venkatramanan, S., & Vullikanti, A. (2020). Mathematical models for covid-19 pandemic: a comparative analysis. Journal of the Indian Institute of Science, 100(4), 793-807.

[7] Demirci, E. (2023). A Novel Mathematical Model of the Dynamics of COVID-19. Gazi University Journal of Science, 36(3), 1302-1309.

[8] Awasthi, A. (2023). A mathematical model for transmission dynamics of COVID-19 infection. The European Physical Journal Plus, 138(3), 285.

[9] Ahmad, S., & Ullah, A., & Al-Mdallal, Q. M., & Khan, H., & Shah, K., & Khan, A. (2020). Fractional order mathematical modeling of COVID-19 transmission. Chaos, Solitons & Fractals, 139, 110256.

[10] Adel, W., &Günerhan, H., &Nisar, K. S., &Agarwal, P., & El-Mesady, A. (2024). Designing a novel fractional order mathematical model for COVID-19 incorporating lockdown measures. Scientific Reports, 14(1), 2926.

[11] Shorten, C., Khoshgoftaar, T. M., & Furht, B. (2021). Deep Learning applications for COVID-19. Journal of big Data, 8(1), 1-54.

[12] Gürsoy, E., & Kaya, Y. (2023). An overview of deep learning techniques for COVID-19 detection: methods, challenges, and future works. Multimedia Systems, 29(3), 1603-1627.

[13] Gokgoz, N. (2023). Forecasting Covid-19 Cases in Türkiye with the Help of LSTM. Black Sea Journal of Engineering and Science, 6(4), 421-425.

[14] Al-Mdallal, Q. M. (2023). Mathematical modeling and simulation of SEIR model for COVID-19 outbreak: A case study of Trivandrum. Frontiers in Applied Mathematics and Statistics, 9, 1124897.

[15] Adomian, G. (2013). Solving frontier problems of physics: the decomposition method (Vol. 60). Springer Science & Business Media.

[16] Adomian, G. (1988). A review of the decomposition method in applied mathematics. Journal of mathematical analysis and applications, 135(2), 501-544.

[17] Adomian, G. (1984). A new approach to nonlinear partial differential equations, J. Math. Anal. Appl., 102 420-434.

[18] Abbaoui, K., & Cherruault, Y. (1999). Convergence of Adomian's method applied to differential equations, Comput. Math. Appl. 102, 77–86.

[19] Luchko, Y. U. R. I. I., & Gorenflo, R. (1999). An operational method for solving fractional differential equations with the Caputo derivatives. Acta Math. Vietnam, 24(2), 207-233.

[20] Podlubny, I. (1999). Fractional Differential Equations, Academic Press, San Diego.

8. THE DARBOUX FRAME AND UMBRELLA MATRIX

Mert Çarboğa **Yusuf Yaylı**

Abstract

In this study, an umbrella matrix was obtained using the Darboux frame of a curve selected on a surface given in the 3-dimensional Euclidean space $\mathbb{E}^3$. With the obtained umbrella matrix, umbrella motion in $\mathbb{E}^3$ was defined.

Keywords. Darboux Matrix, Umbrella Matrix, Infinitesimal Motion

ORCID: 0000-0003-0938-0348
e-mail: mcarboga@ankara.edu.tr

ORCID: 0000-0003-4398-3855
e-mail: yayli@science.ankara.edu.tr

1 Introduction

When studying the differential geometry of surfaces in 3-dimensional Euclidean space, a Darboux frame constructed on such surfaces is of significant importance. In particular, a general helix curve on the surface and the Darboux frame of this curve can yield interesting results.

The umbrella matrices have important applications in geometry, algebra, physics and are defined as follows. Let $\mathbf{1} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$ be given, then the matrix $A \in O(n)$ that satisfies the property $A\mathbf{1} = \mathbf{1}$, is an umbrella matrix, and the set of these matrices is denoted as follows:

$$A(n) = \{A \mid AA^T = I_n,\ A\mathbf{1} = \mathbf{1}\ ,\ \mathbf{1} = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}^T \in \mathbb{R}_1^n\}$$

This definition was proposed made by O. Alisbah in 1976 and was focused on motion geometry by H. Hacısalihoğlu in 1977 with the study [3]. Subsequently, E. Özdamar examined the Lie group and Lie algebra structure of these matrices in the study [5]. Moreover, in the study [4], N. Kuruoğlu generalized this definition for $A \in GL(n, \mathbb{R})$ and $\det(A) \neq 0$ and defined matrix A as a double umbrella matrix in the case

$$A\mathbf{1} = \mathbf{1}\ ,\ \ A^T\mathbf{1} = \mathbf{1}$$

In recent years, Y. Yaylı and M. Çarboğa focused on the geometry of 3-dimensional Dual Umbrella matrices in the study [6] and examined the relationship between these matrices and pairwise comparison matrices in the study [7]. Furthermore, studies on umbrella matrices were published as a booklet in [14], and in the study [13], a transition from the Euclidean umbrella matrix to the Lorentzian umbrella matrix was defined.

The main motivation for this study is the work titled "Umbrella Matrices and Higher Curvatures of a Curve" by Erdoğan Esin in 1986. In this study, the definition of umbrella motion through curves on surfaces was provided using the Cayley equation. Additionally, the relationship between the curve-surface frame and the Darboux matrixis given, and an infinitesimal umbrella motion is obtained. In the study [1], this situation was generalized to every antisymmetric matrix with zero row sums.

In this study, we propose a theorem to generate umbrella motion using a curve chosen on the surfaces in 3-dimensional Euclidean space. An umbrella motion can only be obtained if the chosen curve is a general helix. Additionally, the angular velocities of the frame motion along the curve and the Frenet frame motion were calculated and found to be equal. Furthermore, after obtaining some approximation curves, surface examples are given.

2 Basic Concepts

In this section, some essential concepts are described.

Definition 2.1. Let B be an antisymmetric matrix. The formula

$$A = (I_n - B)^{-1}(I_n + B)$$

is called the Cayley formula. Here, A is an orthogonal matrix, and none of its eigenvalues is -1 [8].

Definition 2.2. In $\mathbb{E}^3$, for $A \in A(n)$, the motion

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

or

$$y = Ax + C$$

is called an umbrella motion in $\mathbb{E}^n$. Here, x, y, and C are matrices of type $(n \times 1)$ [3].

Definition 2.3. Let $y = Ax + C$ be a general motion in $\mathbb{E}^3$. The vectors y and x respectively represent the position vectors of a point P in Σ^n and $\mathbb{E}^n$. For the rotation part of a general motion given by

$$y = A(t)x,$$

the velocity vector at point P is given by

$$\dot{y} = \dot{A}x.$$

Thus, we obtain

$$\dot{y} = \dot{A}A^{-1}y.$$

Here, the matrix

$$W = \dot{A}A^{-1}$$

is called the Darboux matrix of the motion corresponding to A [9].

Definition 2.4. An infinitesimal quantity of the first order ε and an anti-symmetric matrix $[b_{ij}]$ form an infinitesimal matrix as

$$I_3 + \varepsilon[b_{ij}].$$

Here, $\varepsilon \neq 0$ and $\varepsilon^2 = 0$ [11].

Theorem 2.5. Let $\alpha : I \longrightarrow \mathbb{E}^n$ be a curve in $\mathbb{E}^3$ given with parameter s. Let k_i be the i-th curvature function of α and $\{X_1, \ldots, X_n\}$ be the Frenet frame. For $1 \leq i \leq n-1$, the Frenet formulas are given by

$$D_{X_1}X_i = X_i^{'} = -k_{(i-1)g}X_{i-1} + k_{ig}X_{i+1} + II(X_1, X_i)X_n, (k_{0g} = k_{(n-1)g} = 0).$$

These Frenet formulas can be written in matrix form as

$$X^{'} = K(X)X.$$

Here, skew symmetric matrix $K(X)$ is called the curvature matrix of the curve α [10].

3 Darboux Frame and Umbrella Matrices in 3-Dimensional Space

In this section, the equality $\kappa_g = -\kappa_n = \tau_g$ in the curvature matrix

$$K = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix}$$

is considered. The conditions necessary for this equality and the frame motions are investigated to obtain some significant results.
First, let M be a surface. Consider the unit speed curve

$$\alpha : I \subset \mathbb{R} \longrightarrow M$$

on this surface. The Darboux frame $\{T, Y, U\}$ of this curve is given by the equality

$$\begin{bmatrix} T \\ Y \\ U \end{bmatrix}' = \begin{bmatrix} 0 & \kappa\cos\theta & -\kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau+\theta' \\ \kappa\sin\theta & -\tau-\theta' & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ U \end{bmatrix}.$$

Here, for the curvature matrix K, the curvatures are given as

$$\begin{aligned} \kappa_g &= \kappa\cos\theta, \\ \kappa_n &= -\kappa\sin\theta, \\ \tau_g &= \tau+\theta'. \end{aligned}$$

These curvatures are called geodesic curvature, normal curvature, and geodesic torsion of the curve, respectively [12].

In the following theorem, we obtain an umbrella matrix by applying the Cayley transformation to the curvature matrix K.

Theorem 3.1. Let M be a surface, and $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed curve. For the curvature matrix of the curve α

$$K = \begin{bmatrix} 0 & \kappa\cos\theta & -\kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau+\theta' \\ \kappa\sin\theta & -\tau-\theta' & 0 \end{bmatrix},$$

and for the orthogonal matrix obtained from the Cayley transformation

$$A = (I_3 - K)^{-1}(I_3 + K),$$

if $\kappa_g = -\kappa_n = \tau_g$, then A is an umbrella matrix.

Proof. For curvature matrix K, we let

$$\kappa \cos\theta = \kappa \sin\theta = \tau + \theta'. \tag{3.2}$$

From here, we obtain the equality $\theta = \frac{\pi}{4}$ and $\theta' = 0$, and with the equality 3.2, we obtain

$$\tau = \frac{\kappa}{\sqrt{2}}.$$

Thus, we can write the curvature matrix K as

$$K = \begin{bmatrix} 0 & \frac{\kappa}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} \\ -\frac{\kappa}{\sqrt{2}} & 0 & \frac{\kappa}{\sqrt{2}} \\ \frac{\kappa}{\sqrt{2}} & -\frac{\kappa}{\sqrt{2}} & 0 \end{bmatrix}. \tag{3.3}$$

Applying the Cayley transformation to 3.3, we obtain the orthogonal matrix

$$A = \frac{1}{2+3\kappa^2} \begin{pmatrix} -\kappa^2+2 & 2(\kappa^2+\sqrt{2}\kappa) & 2(\kappa^2-\sqrt{2}\kappa) \\ 2(\kappa^2-\sqrt{2}\kappa) & -\kappa^2+2 & 2(\kappa^2+\sqrt{2}\kappa) \\ 2(\kappa^2+\sqrt{2}\kappa) & 2(\kappa^2-\sqrt{2}\kappa) & -\kappa^2+2 \end{pmatrix}.$$

Now, we demonstrate that matrix A is an umbrella matrix. For this reason, it is sufficient to show that for $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \in \mathbb{R}^3_1$,

$$A\mathbf{1} = \mathbf{1}.$$

Here, we simply obtain

$$\begin{aligned} A\mathbf{1} &= \frac{1}{2+3\kappa^2} \begin{bmatrix} 2+3\kappa^2 \\ 2+3\kappa^2 \\ 2+3\kappa^2 \end{bmatrix} \\ A\mathbf{1} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A\mathbf{1} &= \mathbf{1}. \end{aligned}$$

Thus, A is an umbrella matrix. □

3.1 Frame Motions Along the Curve

The Darboux vector of the frame motion $\{T, Y, U\}$ along the curve α is given by

$$W = \frac{\kappa}{\sqrt{2}}T + \frac{\kappa}{\sqrt{2}}Y + \frac{\kappa}{\sqrt{2}}U,$$

and the angular velocity was calculated as $\|W\| = \frac{\sqrt{3}}{\sqrt{2}}\kappa$. Additionally, by considering the Frenet frame $\{T, N, B\}$ along the curve α with $\theta = \int \tau ds$, we obtain

$$\begin{aligned} T &= T, \\ N_1 &= \cos\theta N - \sin\theta B, \\ N_2 &= \sin\theta N + \cos\theta B. \end{aligned}$$

The equation can be written as

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ U \end{bmatrix}.$$

Here, $\kappa_1 = \kappa\cos\theta$ and $\kappa_2 = \kappa\sin\theta$. Thus, the RMF frame $\{T, N_1, N_2\}$ is obtained by the rotation of the Frenet frame $\{T, N, B\}$ around T at angle θ. The Darboux vector of the frame motion is given by

$$\widetilde{W} = -\kappa_2 N_1 + \kappa_1 N_2,$$

and its angular velocity is $\|\widetilde{W}\| = \kappa$. Finally, for the Frenet frame $\{T, N, B\}$ of the curve $\alpha : I \subset \mathbb{R} \longrightarrow M$, we obtain

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The Darboux matrix formed by frame motion is given by

$$\bar{W} = \sqrt{\kappa^2 + \tau^2},$$

and for $\tau = \frac{\kappa}{\sqrt{2}}$, the angular velocity is given by

$$\|\bar{W}\| = \frac{\sqrt{3}}{\sqrt{2}}\kappa.$$

Thus, we obtain the following result:

Corollary 3.4. The angular velocities of the frame motion $\{T, Y, U\}$ along the curve α and the frame motion $\{T, N, B\}$ are equal. These angular velocities are also close to the angular velocity of the RMF frame.

Now, we approximate curve $\alpha \subset M$. For the neighborhood of $t_0 = 0$, the Taylor series expansion of this curve is given by

$$\alpha(t) = \alpha(0) + t\,\alpha'(0) + \frac{t^2}{2!}\,\alpha''(0) + \frac{t^3}{3!}\,\alpha'''(0) + \dots. \tag{3.5}$$

Here, assuming $\alpha(0) = 0$, for the frame $\{T, Y, U\}$, we have

$$\begin{aligned}
\alpha'(0) &= T \\
\alpha''(0) &= \frac{\kappa}{\sqrt{2}}(Y - U) \\
\alpha'''(0) &= -\kappa^2 T + (\frac{\kappa'}{\sqrt{2}} + \frac{\kappa^2}{2})Y + (-\frac{\kappa'}{\sqrt{2}} + \frac{\kappa^2}{2})U
\end{aligned}$$

Substituting these into equation 3.5 and simplifying, we obtain the following:

$$\begin{aligned}
\alpha(t) &= (\frac{6t - \kappa^2 t^3}{6})T + (\frac{3\sqrt{2}\kappa t^2 + (\sqrt{2}\kappa' + \kappa^2)t^3}{12})Y \\
&\quad +(\frac{-3\sqrt{2}\kappa t^2 + (-\sqrt{2}\kappa' + \kappa^2)t^3}{12})U.
\end{aligned}$$

Hence, we obtain the approximation of the curve.

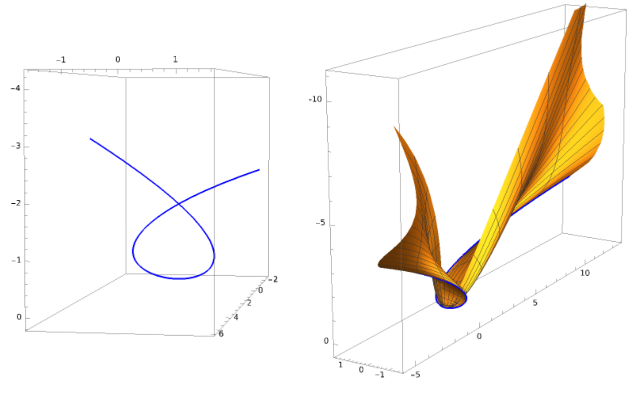
Example 3.6. For the curve $\alpha(t)$, considering $\kappa = t + 1$, we get

$$\alpha(t) = (\frac{6t - t^3}{6})T + (\frac{3\sqrt{2}t^2 + (\sqrt{2}+1)t^3}{12})Y + (\frac{-3\sqrt{2}t^2 + (-\sqrt{2}+1)t^3}{12})U.$$

From the family of surfaces on which this curve is located, one such surface is given by

$$\begin{aligned}
F(t, u) &= (\frac{(6t - t^3)\cos u}{6}, \\
&\quad \frac{(3\sqrt{2}t^2 + (\sqrt{2}+1)t^3)(\sqrt{u}+1)}{12}, \\
&\quad \frac{(-3\sqrt{2}t^2 + (-\sqrt{2}+1)t^3)e^u}{12})
\end{aligned}$$

and the curve-surface pair can be drawn as



If we compute one more derivative of the curve $\alpha(t)$, we obtain

$$\begin{aligned}\beta(t) &= \frac{96t-16\kappa^2t^3-12\kappa\kappa's^4}{96}T\\ &+\frac{24\sqrt{2}\kappa t^2+(8\sqrt{2}\kappa'+8\kappa^2)t^3+(-3\sqrt{2}\kappa^3+6\kappa\kappa'+2\sqrt{2}\kappa'')t^4}{96}Y\\ &+\frac{-24\sqrt{2}\kappa t^2+(-8\sqrt{2}\kappa'+8\kappa^2)t^3+(3\sqrt{2}\kappa^3+6\kappa\kappa'-2\sqrt{2}\kappa'')t^4}{96}U\end{aligned}$$

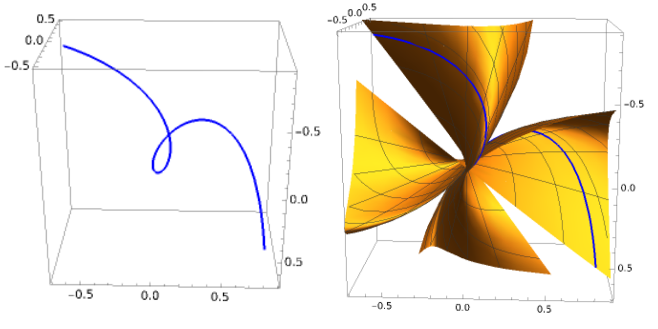
Example 3.7. For the curve $\beta(t)$, considering $\kappa=\cos t+1$, we obtain

$$\beta_1(t)=\frac{96t-64t^3}{96}T+\frac{48\sqrt{2}t^2+32t^3-26\sqrt{2}t^4}{96}Y+\frac{-48\sqrt{2}t^2+32t^3+26\sqrt{2}t^4}{96}U.$$

From the family of surfaces on which this curve lies, one such surface is

$$\begin{aligned}G(t,u) &= (\frac{(96t-64t^3)\cos u}{96},\\ &\quad \frac{(48\sqrt{2}t^2+32t^3-26\sqrt{2}t^4)(u+1)}{96},\\ &\quad \frac{(-48\sqrt{2}t^2+32t^3+26\sqrt{2}t^4)(u+1)}{96})\end{aligned}$$

and the curve-surface pair is drawn as



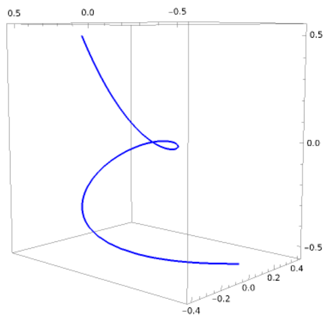
Furthermore, to visualize the curve $\beta(t)$ more simply, considering $\kappa(t)=\sqrt{6}\cos\sqrt{3}t$, we get

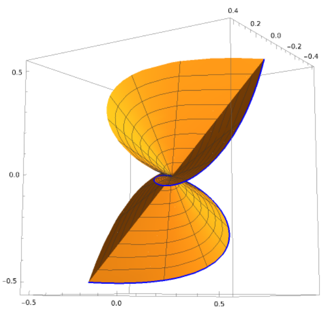
$$\beta_2(t)=(t-t^3)T+\frac{\sqrt{3}t^2+t^3-\sqrt{3}t^4}{2}Y+\frac{-\sqrt{3}t^2+t^3+\sqrt{3}t^4}{2}U.$$

From the family of surfaces on which this curve lies, a conical surface can be given by

$$H(t,u)=((t-t^3)u,\frac{(\sqrt{3}t^2+t^3-\sqrt{3}t^4)u}{2},\frac{(-\sqrt{3}t^2+t^3+\sqrt{3}t^4)u}{2})$$

and the curve-surface equations are respectively given by

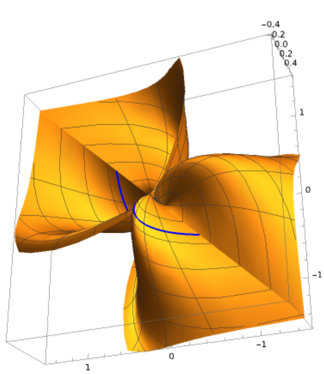




Finally, from the family of surfaces on which the curve $\beta_2(t)$ lies, another surface is given by

$$L(t,u)=((t-t^3)\cos u,\frac{(\sqrt{3}t^2+t^3-\sqrt{3}t^4)(u+1)}{2},\frac{(-\sqrt{3}t^2+t^3+\sqrt{3}t^4)(u+1)}{2}),$$

and the surface equation can be drawn as



4 Infinitesimal Umbrella Motion

In this section, after providing a theorem describing the relationship between the Darboux matrix of the obtained umbrella motion and curvature matrix K, infinitesimal motion is defined.

In [2], the condition that the sum of the rows of the $(n \times n)$-type curvature matrix K is zero is considered, and the relationship between the Darboux matrix of the umbrella motion and the curvature matrix K is given. Additionally, in [1], this situation is generalized to any antisymmetric matrix with zero row sums, providing a different relationship.

In the following theorem, the relationship for the (3×3)-type curvature matrix K is examined.

Theorem 4.1. In $\mathbb{E}^3$, for the umbrella matrix $A = (I_3 - K)^{-1}(I_3 + K)$, there is the relationship

$$W(A) = 2(I_3 - K)^{-1} K' (I_3 + K)^{-1}$$

between the Darboux matrix $W(A)$ of umbrella motion and curvature matrix K.

Theorem 4.2. In $\mathbb{E}^3$, for the umbrella matrix $A = (I_3 - K)^{-1}(I_3 + K)$, the matrix $I_3 + \varepsilon W(A)$ is also an (infinitesimal) umbrella matrix.

Proof. First, we show that the matrix $I_3 + \varepsilon W(A)$ is orthogonal. For this purpose, we must show

$$(I_3 + \varepsilon W(A))(I_3 + \varepsilon W(A))^T = I_3.$$

Since $W(A)^T = -W(A)$, we obtain

$$(I_3 + \varepsilon W(A))(I_3 - \varepsilon W(A)) = I_3^2 - \varepsilon^2 W(A)^2.$$

Considering that $\varepsilon^2 = 0$ for infinitesimal quantity ε, we obtain

$$(I_3 + \varepsilon W(A))(I_3 - \varepsilon W(A)) = I_3.$$

Thus, $I_3 + \varepsilon W(A)$ is an infinitesimal orthogonal matrix. Finally, for $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \in \mathbb{R}_1^3$,

$$(I_3 + \varepsilon W(A))\mathbf{1} = \mathbf{1} + \varepsilon W(A)\mathbf{1},$$

and since $W(A)\mathbf{1} = 0$,

$$(I_3 + \varepsilon W(A))\mathbf{1} = \mathbf{1}.$$

Thus, $I_3 + \varepsilon W(A)$ is an infinitesimal umbrella matrix. □

Theorem 4.3. In $\mathbb{E}^3$, for the umbrella matrix $A = (I_3 - K)^{-1}(I_3 + K)$, the matrix $I_3 + \varepsilon K$ is also an (infinitesimal) umbrella matrix.

Proof. This can be obtained easily as in Theorem 4.2. □

5 Conclusion

In this study, an umbrella matrix on a surface selected in 3-dimensional Euclidean space using a Darboux frame is examined, and the motion of this matrix on the surface is analyzed.

If $\kappa_g = -\kappa_n = \tau_g$, the orthogonal matrix obtained through the Cayley transformation can be considered an umbrella matrix. The angular velocity of the Darboux frame motion along the curve was found to be equal to the angular velocity of the Frenet frame motion and close to the angular velocity of the RMF frame. The approximation curves obtained from the expansion of the Taylor series of the curve and the surfaces on which these curves lie are exemplified. The relationship between the Darboux matrix of the umbrella motion and the curvature matrix is given. Based on this Darboux matrix, the infinitesimal umbrella motion is defined.

We believe that the results obtained have potential applications in field such as geometry, algebra, and kinematics. In future studies, the relationships between these matrices and other geometric structures and their generalizations in different spaces will be examined.

References

[1] Çarboğa M., Yaylı Y., "Geometric applications and kinematics of umbrella matrices." Korean J. of Math., vol. 31, pp. 295-303, 2023.

[2] Esin E., "Umbrella matrices and higher curvatures of a curve." Commu. Facul. Sci. Uni. Ankara Series A1 Math. and Stat., vol. 35, pp. 28–34, 1986.

[3] Hacısalihoğlu H., "Umbrella motions." J. Facul. Sci. Karadeniz Tech. Uni., vol. 1, pp. 29-40, 1977.

[4] Kuruoglu N., "On the lie group of umbrella matrices." Commu. Facul. Scie. Uni. Ankara Series A1 Math. and Stat., vol. 32, pp. 132–144, 1983.

[5] Ozdamar E., "The lie group of umbrella matrices and differential geometry." Doctoral thesis Ankara Uni., 1977.

[6] Çarboğa M., Yaylı Y., "Geometry Of Dual Umbrella Matrices." Advanced Stud. Euro-Tbilisi Math. J., vol. 16, pp. 77-87, 2023.

[7] Çarboğa M., Yaylı Y., "The Lie Group of Umbrella Matrices and Pairwise Comparisons Matrices." Acta Uni. Apu., vol. 74, pp. 1-10, 2023.

[8] McCarthy J. M., "An Introduction to Theoretical Kinematics." MIT Press Cambridge, 1990.

[9] Bottema O., Roth B., "Theoretical kinematics." Courier Corporation ABD, 1990.

[10] Guggenheimer W., ”Differential geometry.” Courier Corporation ABD, 2012.

[11] Courant R., Hilbert D., ”Methods of mathematical physics: partial differential equations.” John Wiley-Sons Singapore, 2008.

[12] Hacısalihoğlu H., ”Yüksek Diferansiyel Geometriye Giriş.” Fırat Üni. Yayınları, 1980.

[13] Yüca G., Yaylı Y., ”Homothetic motions and dual transformations.” Erciyes Üniversitesi Fen Bilimleri Enstitüsü Fen Bilimleri Dergisi, vol. 37, no. 1, pp. 194-205, 2021.

[14] Çarboğa M., Yaylı Y., ”Differential Geometry of Umbrella Matrices.” Int. Geo. Sym. in Memory of Prof. Erdogan ESIN’s Proc. Book, 2023.

9. $\mathcal{I}_2 - \alpha\beta$-STATISTICAL CONVERGENCE FOR DOUBLE SEQUENCES DEFINED BY ORLICZ FUNCTION

Sevda Yıldız **Nilay Şahin Bayram**

Abstract

The present paper aims to give the notions of strong $\mathcal{I}_2 - \alpha\beta$-summability and $\mathcal{I}_2 - \alpha\beta$-statistical convergence with respect to an Orlicz function. Based on these new notions of convergence, new double sequence spaces and some of their properties are given. Furthermore, certain inclusion relations are being analyzed. Finally, it is remarked that the results can be obtained for modulus function under proper choices.

Keywords. Power series method, double sequences, statistical convergence, Orlicz function.

ORCID: 0000-0002-4730-2271
e-mail: sevdaorhan@sinop.edu.tr

ORCID: 0000-0003-3263-8589
e-mail: nsbayram@baskent.edu.tr

1 Introduction

Statistical convergence, initially proposed by Steinhaus [29] and Fast [14] for real sequences, transcended the boundaries of classical convergence. This notion has seen various extensions and applications across different spaces, following significant contributions by Fridy [12] and Fridy and Orhan [13] and Šalát [27]. In recent years, numerous authors (e.g., [18, 19, 30, 31, 32, 34]) have further expanded the idea of statistical convergence. Additionally, Kostyrko et al. [20] (also independently by Nuray and Ruckle [25]) introduced $\mathcal{I}$-convergence of sequences, where $\mathcal{I}$ represents an ideal of subsets of natural numbers. They further extended this concept to sequences of real functions. Subsequently, Das et al. [8] generalized this convergence to double sequences, and Dündar and Altay [11] explored various ideal convergence notions for double sequences of real-valued functions. Building on these advancements, Aktuğlu [1] pioneered a powerful variant of statistical convergence, termed $\alpha\beta$-statistical convergence of order γ. This concept was shown to be a significant extension of both ordinary and statistical convergences. Recognizing the importance of double sequences, Altundağ and Sözbir [2] introduced $\alpha\beta$-statistical convergence and strong $\alpha\beta$-summability for double sequences, investigating the interplay between these novel concepts. Recently, Savaş and Das [28] introduced the innovative concept of $\mathcal{I}$-statistical convergence, merging statistical convergence with $\mathcal{I}$-convergence. Belen and Yıldırım [5] generalized this notion to double sequences, leading to new summability methods that unify the notions of statistical convergence. Most recently, Ghosal and Mandal [15] introduced $\mathcal{I} - \alpha\beta$-statistical convergence for single sequences, representing a further generalization of $\mathcal{I}$-statistical convergence.

Several convergence techniques, including statistical convergence, strong convergence, and strong convergence with respect to Orlicz functions, hold significance in mathematical analysis. In recent years, numerous investigations have focused on these concepts and their interrelations (refer to, for instance, [16],[17],[21], [30], [34]).

Building upon the aforementioned research, this article introduces novel concepts of strong $\mathcal{I}_2 - \alpha\beta$-summability and $\mathcal{I}_2 - \alpha\beta$-statistical convergence for double sequences in the context of Orlicz functions. It further outlines new double sequence spaces and explains their properties derived from these recently introduced convergence concepts. Additionally, this study examines inclusion relations among these concepts. It is noted that selecting appropriate parameters enables the derivation of results for the modulus function.

For our investigation, the followings are requisite. First recall the main concepts of convergence methods for double sequences.

Definition 1.1. [26] A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim's sense (P-convergent) if

$$\forall \varepsilon > 0,\ \exists J = J(\varepsilon), \forall i, j > J,\ |x_{ij} - L| < \varepsilon.$$

It is denoted by $P - \lim_{i,j} x_{ij} = L$.

Definition 1.2. [26] A double sequence $x = (x_{ij})$ is called bounded if

$$\exists K > 0, \forall (i, j) \in \mathbb{N}^2,\ |x_{ij}| \leq K, \text{ i.e.,}$$

if $\|x\|_{(\infty,2)} = \sup_{i,j} |x_{ij}| < \infty$ and denote the set by l_∞^2.

Definition 1.3. [24] The number sequence $x = (x_{ij})$ is statistically convergent to L provided that, for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{|\{i \leq m, j \leq n : \ |x_{ij} - L| \geq \varepsilon\}|}{mn} = 0,$$

in that case we write $st_2 - \lim_{i,j} x_{ij} = L$.

Now let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

C_1 : $\alpha,\ \beta$ are both non-decreasing,

C_2 : $\beta(n) \geq \alpha(n)$,

C_3 : $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$,

and let Λ denote the set of pairs (α, β) satisfying C_1, C_2 and C_3[1].

Throughout the paper, let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \Lambda$.

Definition 1.4. [2]A double sequence $x = (x_{ij})$ is said to be $\alpha\beta$-statistically convergent to L, if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{\left|D_m^{\alpha_1\beta_1}\right| \left|D_n^{\alpha_2\beta_2}\right|} \left|\left\{(i, j),\ i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : |x_{ij} - L| \geq \varepsilon\right\}\right| = 0$$

which is denoted by $st_2^{\alpha\beta} - \lim_{i,j} x_{ij} = L$, where $D_m^{\alpha_1\beta_1}$ and $D_n^{\alpha_2\beta_2}$ are closed intervals $[\alpha_1(m), \beta_1(m)]$ and $[\alpha_2(n), \beta_2(n)]$, respectively.

Definition 1.5. [2]A double sequence $x = (x_{ij})$ is said to be strongly $\alpha\beta$-summable to L, if for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{\left|D_m^{\alpha_1\beta_1}\right| \left|D_n^{\alpha_2\beta_2}\right|} \sum_{i \in D_m^{\alpha_1\beta_1}} \sum_{j \in D_n^{\alpha_2\beta_2}} |x_{ij} - L| = 0.$$

Now we recall some notation and terminology of an ideal. Kostyrko, Salat and Wilczynski, have defined $\mathcal{I}$-convergence using the ideal $\mathcal{I}$ ([20]). This type of convergence can be seen as a general form of statistical convergence. Let a class $\mathcal{I}$ of subsets of X, a non-empty set, is called an *ideal* in X iff (i) $\emptyset\in\mathcal{I}$, (ii) $A,B\in\mathcal{I}$ implies $A\cup B\in\mathcal{I}$ and (iii) for each $A\in\mathcal{I}$ and $B\subset A$ we have $B\in\mathcal{I}$. If $\{x\}\in\mathcal{I}$ for each $x\in X$ then an ideal called *admissible.* If $\mathcal{I}$ is a *non-trivial ideal* in X (i.e. $X\notin\mathcal{I},\mathcal{I}\neq\{\emptyset\}$) then the family of sets $\mathcal{F}=\{U\subset X:(\exists A\in\mathcal{I})\,(U=X\setminus A)\}$ is a *filter* in X and we call such an filter, the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}^2$ is called s*trongly admissible* if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in\mathbb{N}$. It seems obvious that a strongly admissible ideal is admissible also. Let

$$\mathcal{I}_2^0=\left\{B\subset\mathbb{N}^2:(\exists m\,(B)\in\mathbb{N})\,(i,j\geq m\,(B)\Rightarrow(i,j)\notin B)\right\},$$

then $\mathcal{I}_2^0$ is a non-trivial strongly admissible ideal ([8]) and clearly $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0\subset\mathcal{I}_2$.

For the rest of the paper, we use $\mathcal{I}_2$ as a non-trivial strongly admissible ideal on $\mathbb{N}^2$.

Definition 1.6. Let $x=(x_{ij})$ be a double sequence. We introduce the following double sequence space

$$w\left(\mathcal{I}_2^{\alpha\beta}\right)=\left\{x=(x_{ij}):\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\times\right.\right.$$
$$\left.\left.\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}|x_{ij}-L|\geq\varepsilon\right\}\in\mathcal{I}_2,\text{ for some }L\right\}.$$

If $x\in w\left(\mathcal{I}_2^{\alpha\beta}\right)$ we say that x is strong $\mathcal{I}_2-\alpha\beta$-summable to L and also, if $L=0$ then we write $w_0\left(\mathcal{I}_2^{\alpha\beta}\right)$.

Definition 1.7. We introduce the following double sequence space,

$$st\left(\mathcal{I}_2^{\alpha\beta}\right)=\left\{x=(x_{ij}):\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\times\right.\right.$$
$$\left.\left.\left|\left\{(i,j)\,,i\in D_m^{\alpha_1\beta_1}\text{ and }j\in D_n^{\alpha_2\beta_2}:|x_{ij}-L|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2\right\}.$$

In this case we say that x is $\mathcal{I}_2-\alpha\beta$-statistically convergent to L and we write $x\in st\left(\mathcal{I}_2^{\alpha\beta}\right)$. Also, we say $st\left(\mathcal{I}_2^{\alpha\beta}\right)-\lim_{i,j}x_{ij}=L$.

In other words, we can write that $\chi_{S(x-Le;\varepsilon)}$ is contained in $w\left(\mathcal{I}_2^{\alpha\beta}\right)$ for every $\varepsilon > 0$ where $\chi_{S(x;\varepsilon)}$ is the characteristic function of the set

$$S(x;\varepsilon) = \left\{(i,j) \in \mathbb{N}^2 : |x_{ij}| \geq \varepsilon\right\}.$$

This definition also includes the following special cases:

(i) If we take $\alpha_1(m) = 1, \beta_1(m) = m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = 1, \beta_2(n) = n$ for all $n \in \mathbb{N}$, then $\mathcal{I}_2 - \alpha\beta$-statistical convergence is reduced to $\mathcal{I}_2-$statistical convergence introduced in [5].

(ii) Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1,\ \lambda_1 = 1,$$
$$\mu_{n+1} \leq \mu_n + 1,\ \mu_1 = 1.$$

Then in the case of $\alpha_1(m) = m - \lambda_m + 1$, $\beta_1(m) = m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = n - \mu_n + 1$, $\beta_1(n) = n$ for all $n \in \mathbb{N}$, $\mathcal{I}_2 - \alpha\beta$-statistical convergence is reduced to $\mathcal{I}_2 - (\lambda, \mu)$-statistical convergence introduced in [5].

(iii) Recall that a double lacunary sequence $\theta_{r,s} = \{(k_r, l_s)\}$, which means there exist two increasing of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$
$$l_0 = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

If we take $\alpha_1(m) = k_{m-1} + 1$, $\beta_1(m) = k_m$ for all $m \in \mathbb{N}$ and $\alpha_2(n) = l_{n-1} + 1$, $\beta_1(n) = l_n$ for all $n \in \mathbb{N}$, then $\mathcal{I}_2 - \alpha\beta$-statistically convergence of double sequence is reduced to $\mathcal{I}_2$-lacunary statistical convergence of double sequence introduced in [22].

Recall that F is called the Orlicz function whenever $F : [0,\infty) \to [0,\infty)$ is a function such that it is continuous, nondecreasing and convex with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$ ([21]). If the convexity condition of Orlicz function F is replaced by $F(x+y) \leq F(x) + F(y)$, then F is called modulus function (see, [23]).

To establish certain results, we require the following property of an Orlicz function. If there is a constant $H > 0$ such that, for all $u > 0$,

$$F(2u) \leq HF(u),$$

we say that the Orlicz function F satisfies the $\triangle_2$-condition. This condition is equivalent to the condition

$$F(tu) \leq HtF(u)$$

for all $u > 0$ and for $t > 1$ (see, e.g., [21]).

2 Strong $\mathcal{I}_2 - \alpha\beta$-Summable and $\mathcal{I}_2 - \alpha\beta$-Statistical Convergence via Orlicz Function

In this section, our primary findings will be thoroughly discussed.

Definition 2.1. Let F be an Orlicz function. We introduce the following double sequence space

$$w\left(\mathcal{I}_2^{\alpha\beta}, F\right) = \left\{ x = (x_{ij}) : \left\{ (m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right| \left|D_n^{\alpha_2\beta_2}\right|} \times \right.\right.$$

$$\left.\left. \sum_{i \in D_m^{\alpha_1\beta_1}} \sum_{j \in D_n^{\alpha_2\beta_2}} F\left(|x_{ij} - L|\right) \geq \varepsilon \right\} \in \mathcal{I}_2, \text{ for some } L \right\}.$$

If $x \in w\left(\mathcal{I}_2^{\alpha\beta}, F\right)$ we say that x is strong $\mathcal{I}_2 - \alpha\beta$-summable to L with respect to an Orlicz function F and also, if $L = 0$ then we write $w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$.

Remark 2.2. If we take $F(x) = |x|$, then $\mathcal{I}_2 - \alpha\beta$ -summability via Orlicz function coincide with $\mathcal{I}_2 - \alpha\beta$- summability. Thus $\mathcal{I}_2 - \alpha\beta$-summability via Orlicz function is a generalization of $\mathcal{I}_2 - \alpha\beta$-summability.

Theorem 2.3. Given any Orlicz function satisfying $\triangle_2$-condition we have the inclusions

$$w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \subset w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$$

and

$$w\left(\mathcal{I}_2^{\alpha\beta}\right) \subset w\left(\mathcal{I}_2^{\alpha\beta}, F\right).$$

Proof. It is enough to prove $w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \subset w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. Now let $x = (x_{ij}) \in w_0\left(\mathcal{I}_2^{\alpha\beta}\right)$ and F be an Orlicz function satisfying $\triangle_2$-condition. From the right continuity of F at zero, for a given $\varepsilon > 0$, there exists η with $0 < \eta < 1$ such that $F(u) < \varepsilon$ whenever $0 \leq u < \eta$. Then we may write that

$$
\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\\
&=\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ \left|x_{ij}\right|<\eta}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\\
&+\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ \left|x_{ij}\right|\geq\eta}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\\
&<\varepsilon+\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ \left|x_{ij}\right|\geq\eta}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right).
\end{aligned}
$$

Since $0<\eta<1$, we get for every $i,j\in\mathbb{N}$, that

$$
\left|x_{ij}\right|<\frac{\left|x_{ij}\right|}{\eta}<1+\frac{\left|x_{ij}\right|}{\eta}.
$$

Also, since F is an Orlicz function satisfying $\triangle_2$-condition, we observe that

$$
\begin{aligned}
F\left(\left|x_{ij}\right|\right)&\leq F\left(1+\frac{\left|x_{ij}\right|}{\eta}\right)=F\left(2\frac{1}{2}+2\frac{\left|x_{ij}\right|}{2\eta}\right)\\
&\leq\frac{1}{2}F\left(2\right)+\frac{1}{2}F\left(\frac{2\left|x_{ij}\right|}{\eta}\right)\\
&\leq\frac{1}{2}F\left(2\right)+\frac{H\left|x_{ij}\right|}{2\eta}F\left(2\right)<\left(1+H\right)F\left(2\right)\frac{\left|x_{ij}\right|}{\eta}.
\end{aligned}
$$

Then combining the above results, we obtain that

$$
\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\\
&<\varepsilon+\frac{\left(1+H\right)}{\eta}F\left(2\right)\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}\left|x_{ij}\right|.
\end{aligned}
$$

From the hypothesis $x = (x_{ij}) \in w_0\left(\mathcal{I}_2^{\alpha\beta}\right)$ and also given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$, we see that

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}|\right) \geq r\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} |x_{ij}| \geq \frac{(r-\varepsilon)\eta}{(1+H)F(2)}\right\} \in \mathcal{I}_2$$

which completes the proof. □

Lemma 2.4. Let F be an Orlicz function satisfying $\triangle_2$-condition. Then $w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$ is an ideal in l_∞^2.

Proof. Let $x \in w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$ and $z \in l_\infty^2$ we show that $xz \in w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. Since $z \in l_\infty^2$, there is $K_1 > 1$ so that $\|z\|_{(\infty,2)} \leq K_1$. Due to F is nondecreasing and satisfies $\triangle_2$-condition we have

$$F\left(|x_{ij}z_{ij}|\right) \leq F\left(K_1|x_{ij}|\right) \leq HK_1F\left(|x_{ij}|\right), \ (H > 0).$$

By hypothesis

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}|\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

so we get that

$$\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}z_{ij}|\right)$$
$$\leq HK_1\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}|\right)$$

from which we immediately conclude that

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}z_{ij}|\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

This proves the result. □

As in above lemma one can observe that $w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l_\infty^2$ is an ideal in l_∞^2. On the other hand we get the following:

Lemma 2.5. Let F be an Orlicz function satisfying $\triangle_2$-condition, then $w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2$ is a closed ideal in l_∞^2. Also, note that, $w_0\left(\mathcal{I}_2^{\alpha\beta}\right)\cap l_\infty^2$ is a closed ideal in l_∞^2.

Proof. It is enough to prove $w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2$ is closed. Let $x\in\overline{w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2}$. Then, there exists $x^{kl}:=x_{ij}^{kl}\in w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2$ with $x^{kl}\to x\in l_\infty^2$. Then given $\varepsilon>0$, we get

$$\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\\
&=\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}-x_{ij}^{kl}+x_{ij}^{kl}\right|\right)\\
&\le\frac{1}{2}\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(2\left|x_{ij}-x_{ij}^{kl}\right|\right)\\
&+\frac{1}{2}\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(2\left|x_{ij}^{kl}\right|\right)\\
&\le\frac{1}{2}HF\left(\varepsilon\right)+\frac{1}{2}H\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}^{kl}\right|\right)\\
&=\varepsilon^{'}+\frac{1}{2}H\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}^{kl}\right|\right),
\end{aligned}$$

where $\varepsilon^{'}=\frac{1}{2}HF\left(\varepsilon\right)$. Given $r>0$, choose $\varepsilon^{'}>0$ such that $\varepsilon^{'}<r$, we see that

$$\begin{aligned}
&\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\ge r\right\}\\
&\subset\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}^{kl}\right|\right)\ge\frac{2\left(r-\varepsilon^{'}\right)}{H}\right\}\in\mathcal{I}_2
\end{aligned}$$

Since $x_{ij}^{kl}\in w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2$, we can get

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}}F\left(\left|x_{ij}\right|\right)\ge r\right\}\in\mathcal{I}_2$$

which gives $x\in w_0\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2$. The proof is completed. □

Lemma 2.6. Let M be an ideal in l^2_∞ and let $x \in l^2_\infty$. Then x is in the closure of M in ℓ^2_∞ iff $\chi_{S(x;\varepsilon)} \in M$ for all $\varepsilon > 0$.

Proof. The proof can be easily obtained by the same technique as in [6]. □

Theorem 2.7. Let F be an Orlicz function that satisfies $\triangle_2$-condition. Then

$$w\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty = w\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l^2_\infty.$$

Proof. Observe that is enough to prove that $w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty = w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l^2_\infty$. By Theorem 2.3, we have that $w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l^2_\infty \subset w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty$.

We now prove the opposite inclusion. Note that

$$\begin{aligned} &\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\chi_{S(x;\varepsilon)}(i,j)\right) \\ &= F(1) \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} \chi_{S(x;\varepsilon)}(i,j) \end{aligned} \tag{2.8}$$

Let $x \in w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty$ and $\varepsilon > 0$. Now define a sequence $z = (z_{ij}) \in l^2_\infty$ by $z_{ij} = \dfrac{1}{x_{ij}}$ if $|x_{ij}| \geq \varepsilon$ and $z_{ij} = 0$ otherwise. Since $xz = \chi_{S(x;\varepsilon)}$ and $\chi_{S(x;\varepsilon)} \in w_0\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty$, we have that

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\chi_{S(x;\varepsilon)}(i,j)\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Thanks to (2.8) that

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \sum_{i\in D_m^{\alpha_1\beta_1}} \sum_{j\in D_n^{\alpha_2\beta_2}} \chi_{S(x;\varepsilon)}(i,j) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Lemma 2.5 and Lemma 2.6 yield that $x \in w_0\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l^2_\infty$ which completes the proof. □

Theorem 2.9. Let F be an Orlicz function. Then, we have
(i) $w\left(\mathcal{I}_2^{\alpha\beta}, F\right) \subseteq st\left(\mathcal{I}_2^{\alpha\beta}\right)$, (ii) $st\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l^2_\infty \subseteq w\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l^2_\infty$.

Proof. (i) Let $x \in w\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. Then

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}-L|\right) \geq \varepsilon\right\} \in \mathcal{I}_2,$$

for some L.For every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}-L|\right)\\
&= \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{\substack{j\in D_n^{\alpha_2\beta_2}\\ |x_{ij}-L|<\varepsilon}} F\left(|x_{ij}-L|\right)\\
&+ \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{\substack{j\in D_n^{\alpha_2\beta_2}\\ |x_{ij}-L|\geq\varepsilon}} F\left(|x_{ij}-L|\right)\\
&\geq \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{\substack{j\in D_n^{\alpha_2\beta_2}\\ |x_{ij}-L|\geq\varepsilon}} F\left(|x_{ij}-L|\right)\\
&\geq F(\varepsilon)\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} \chi_{S(x-Le;\varepsilon)}(i,j),
\end{aligned}$$

this implies that $x \in st\left(\mathcal{I}_2^{\alpha\beta}\right)$.

(ii) Let $x \in st\left(\mathcal{I}_2^{\alpha\beta}\right) \cap l_\infty^2$. Since $x \in l_\infty^2$, we can write, for all $(i,j) \in \mathbb{N}_0^2$, that $|x_{ij}-L| < |x_{ij}| + |L| = K$. Then

$$\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(|x_{ij}-L|\right)\\
&= \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{\substack{j\in D_n^{\alpha_2\beta_2}\\ |x_{ij}-L|<\varepsilon}} F\left(|x_{ij}-L|\right)\\
&+ \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{\substack{j\in D_n^{\alpha_2\beta_2}\\ |x_{ij}-L|\geq\varepsilon}} F\left(|x_{ij}-L|\right)\\
&\leq F(\varepsilon) + F(K)\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} \chi_{S(x-Le;\varepsilon)}(i,j).
\end{aligned}$$

Thanks to continuity of F and since $F(0)=0$, we get $x\in w\left(\mathcal{I}_2^{\alpha\beta},F\right)$.
□

Corollary 2.10. Let F be an Orlicz function. Then

$$st\left(\mathcal{I}_2^{\alpha\beta}\right)\cap l_\infty^2=w\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2.$$

We will proceed with the definition that follows, providing us with a new sequence space:

Definition 2.11. Let F be an Orlicz function. We introduce the following double sequence space

$$st\left(\mathcal{I}_2^{\alpha\beta},F\right)=\left\{x=(x_{ij}):\left\{(m,n)\in\mathbb{N}^2:\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\times\right.\right.$$
$$\left.\left.\left|\left\{(i,j),i\in D_m^{\alpha_1\beta_1}\text{ and }j\in D_n^{\alpha_2\beta_2}:F(|x_{ij}-L|)\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2\right\}.$$

If $x\in st\left(\mathcal{I}_2^{\alpha\beta},F\right)$ we say that x is $\mathcal{I}_2-\alpha\beta$-statistically convergent with respect to an Orlicz function F. Also, we denote this limit by $st\left(\mathcal{I}_2^{\alpha\beta}\right)-F-\lim_{i,j}x_{ij}=L$.

In view of the above definition, we obtain the following propositions:

Proposition 2.12. If $x\in st\left(\mathcal{I}_2^{\alpha\beta},F\right)$, then the limit of x is uniquely determined.

Proposition 2.13. Let $x=(x_{ij})$ and $y=(y_{ij})$ be two double sequences. If $st\left(\mathcal{I}_2^{\alpha\beta},F\right)-\lim_{i,j}x_{ij}=L_1$ and $st\left(\mathcal{I}_2^{\alpha\beta},F\right)-\lim_{i,j}y_{ij}=L_2$, then the following statements hold:

$$\begin{aligned}(i)\quad & st\left(\mathcal{I}_2^{\alpha\beta},F\right)-\lim_{i,j}(x_{ij}+y_{ij})=L_1+L_2,\\(ii)\quad & st\left(\mathcal{I}_2^{\alpha\beta},F\right)-\lim_{i,j}cx_{ij}=cL_1,\ (c\in\mathbb{R}).\end{aligned}$$

Then, we introduce the following theorem.

Theorem 2.14. Let F be an Orlicz function. Then, we have $w\left(\mathcal{I}_2^{\alpha\beta},F\right)\subseteq st\left(\mathcal{I}_2^{\alpha\beta},F\right)$. Also, $st\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2(F)\subseteq w\left(\mathcal{I}_2^{\alpha\beta},F\right)\cap l_\infty^2(F)$ for an Orlicz function which satisfies the $\triangle_2$-condition where

$$l_\infty^2(F):=\left\{x=(x_{ij}):\exists K>0,\forall(i,j)\in\mathbb{N}^2,F(|x_{ij}|)\leq K\right\}.$$

Proof. Let $x \in w\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. For every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\left|x_{ij}-L\right|\right)\\
&\geq \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ F(|x_{ij}-L|)\geq\varepsilon}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\left|x_{ij}-L\right|\right)\\
&\geq \varepsilon\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\chi_{S(x-Le;\varepsilon)}\left(i,j\right)\right),
\end{aligned}$$

this implies that $x \in st\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. Therefore $w\left(\mathcal{I}_2^{\alpha\beta}, F\right) \subseteq st\left(\mathcal{I}_2^{\alpha\beta}, F\right)$.

Now, let $x \in st\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l_\infty^2\left(F\right)$. Since $x \in l_\infty^2\left(F\right)$ and also, from the convexity of F and the $\triangle_2$-condition, we can write, for all $(i,j) \in \mathbb{N}_0^2$, that

$$\begin{aligned}
F\left(\left|x_{ij}-L\right|\right) &< \frac{1}{2}F\left(2\left|x_{ij}\right|\right)+\frac{1}{2}F\left(2\left|L\right|\right) \leq \frac{1}{2}H_1F\left(\left|x_{ij}\right|\right)+\frac{1}{2}H_2F\left(\left|L\right|\right)\\
&\leq \frac{1}{2}\left\{H_1K+H_2F\left(\left|L\right|\right)\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\left|x_{ij}-L\right|\right)\\
&= \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ F(|x_{ij}-L|)<\varepsilon}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\left|x_{ij}-L\right|\right)\\
&+ \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{\substack{i\in D_m^{\alpha_1\beta_1}\\ F(|x_{ij}-L|)\geq\varepsilon}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\left|x_{ij}-L\right|\right)\\
&\leq \varepsilon+\frac{1}{2}\left\{H_1K+H_2F\left(\left|L\right|\right)\right\}\frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|}\sum_{i\in D_m^{\alpha_1\beta_1}}\sum_{j\in D_n^{\alpha_2\beta_2}} F\left(\chi_{S(x-Le;\varepsilon)}\left(i,j\right)\right),
\end{aligned}$$

which yields $x \in w\left(\mathcal{I}_2^{\alpha\beta}, F\right)$. □

We will now investigate the relationship between $\alpha\beta$-statistical convergence and $\mathcal{I}_2-\alpha\beta$-statistical convergence with respect to an Orlicz function for double sequences.

Theorem 2.15. Let $F : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. For any double sequence (x_{ij}), $st_2^{\alpha\beta}-F-\lim_{i,j} x_{ij} = L$ implies $st\left(\mathcal{I}_2^{\alpha\beta}\right)-F-\lim_{i,j} x_{ij} = L$.

Proof. Let $st_2^{\alpha\beta} - F - \lim_{i,j} x_{ij} = L$. Then for every $\varepsilon > 0$,

$$P-\lim_{m,n} \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \left|\left\{(i,j)\,,\ i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : F\left(|x_{ij} - L|\right) \geq \varepsilon\right\}\right| = 0.$$

Hence for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{\left|D_m^{\alpha_1\beta_1}\right|\left|D_n^{\alpha_2\beta_2}\right|} \left|\left\{(i,j)\,, i \in D_m^{\alpha_1\beta_1} \text{ and } j \in D_n^{\alpha_2\beta_2} : F\left(\left|x_{ij} - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\}$$

is a finite set and therefore belongs to $\mathcal{I}_2$, as $\mathcal{I}_2$ is a strongly admissible ideal. Thus, $st\left(\mathcal{I}_2^{\alpha\beta}\right) - F - \lim_{i,j} x_{ij} = L$. □

3 Concluding Remarks

As mentioned earlier in this paper, if the convexity condition of Orlicz function F is replaced by $F(x+y) \leq F(x) + F(y)$, then F is called a modulus function (see, [23]). Hence, with considering slight modification in the proofs, it can be easily find that

$$st\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l_\infty^2 = w\left(\mathcal{I}_2^{\alpha\beta}, f\right) \cap l_\infty^2,$$

where f is the modulus function. Let F be an Orlicz and f be a modulus function. Then

$$w\left(\mathcal{I}_2^{\alpha\beta}, f\right) \cap l_\infty^2 = w\left(\mathcal{I}_2^{\alpha\beta}, F\right) \cap l_\infty^2.$$

References

[1] Aktuğlu H. (2014) , Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence, J. Comput. Appl. Math. 259A, 174–181.

[2] Altundağ, S., & Sözbir, B. (2019), Korovkin Type Approximation Theorem for Functions of Two Variables Through $\alpha\beta$-Statistical Convergence. Journal of Mathematical Sciences and Modelling, 2(3), 198-204.

[3] Baron S., Stadtmüller U. (1997), Tauberian theorems for power series methods applied to double sequences, J. Math. Anal. Appl., 211(2), 574–589.

[4] Bayram, N. Ş., & Yildiz, S. (2022), Approximation by statistical convergence with respect to power series methods. Hacettepe Journal of Mathematics and Statistics, 1-13.

[5] Belen, C., & Yildirim, M. (2012). On generalized statistical convergence of double sequences via ideals. Annali Dell'universita'Di Ferrara, 58, 11-20.

[6] Connor J. (1989), On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32; 194-198.

[7] Das, P., Savas, E., & Ghosal, S. K. (2011). On generalizations of certain summability methods using ideals. Applied mathematics letters, 24(9), 1509-1514.

[8] Das, P., & Kostyrko, P., & Wilczyński, W., & Malik, P. (2008). I and I*-convergence of double sequences. Mathematica Slovaca, 58(5), 605-620.

[9] Demirci, K., Dirik, F., & Yıldız, S. (2022), Approximation via equi-statistical convergence in the sense of power series method. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 116(2), 65.

[10] Demirci K. (1996), Strong A-summability and A-statistical convergence. Indian Journal Pure and Applied Mathematics; 27: 589-593.

[11] Dündar, E., & Altay, B. (2015). I2-convergence of double sequences of functions. Electronic Journal of Mathematical Analysis and Applications, 3(1), 111-121.

[12] Fridy, J.A. (1993). Statistical limit points, Proc. Am. Math. Soc. 118, 1187–1192.

[13] Fridy, J.A., & Orhan, C. (1993). Lacunary statistical convergence, Pac. J. Math. 160, 43–51.

[14] Fast, H. (1951). Sur la convergence statistique, Colloq. Math. 2, 241–244.

[15] Ghosal S., & Mandal S. (2022), Rough weighted $\mathcal{I}$-$\alpha\beta$-statistical convergence in locally solid Riesz spaces, J. Math. Anal. Appl. 506; 125681.

[16] Gülfırat M., & Şahin Bayram N. (2023), On uniform statistical convergence, Georgian Mathematical Journal, 30 (6), 873-882.

[17] Gülfırat, M., & Bayram, N. Ş. (2021), Multipliers and I-core for sequences, Turkish Journal of Mathematics, 45 (3), 1310-1318.

[18] Khan, M. K., & Orhan, C. (2007), Matrix characterization of A-statistical convergence. Journal of Mathematical Analysis and Applications; 335: 406-417.

[19] Khan M. K., & Orhan C. (2010), Characterizations of strong and statistical convergence. Publicationes Mathematicae Debrecen; 76 (1-2): 77-88.

[20] Kostyrko P., & Šalát T., & Wilczyński W. (2000). $\mathcal{I}$ -convergence. Real Analysis Exchange, 26(2): 669-686.

[21] Krasnoselskii M. A., & Rutisky Y. B. (1961), Convex Function and Orlicz Spaces, Groningen, Netherlands.

[22] Kumar, S., & Kumar, V., & Bhatia, S. S. (2013). On ideal version of lacunary statistical convergence of double sequences. General Mathematics Notes, 17(1), 32-44.

[23] Maddox I. J. (1986), Sequence spaces defined by modulus, Math. Proc. Camb. Phil. Soc. 100; 161-166.

[24] Moricz F. (2004), Statistical convergence of multiple sequences. Arch. Math. (Basel) **81**, 82-89.

[25] Nuray, F., & Ruckle, W. H. (2000). Generalized statistical convergence and convergence free spaces. Journal of Mathematical Analysis and Applications, 245(2), 513-527.

[26] Pringsheim A. (1900), Zur theorie der zweifach unendlichen zahlenfolgen. Math. Ann. **53**, 289-321.

[27] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139–150.

[28] Savas, E., & Das, P. (2011). A generalized statistical convergence via ideals, App. Math. Lett. 24 (6) 826–830.

[29] Steinhaus H. (1951). Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2, 73–74.

[30] Bayram, N. Ş. (2022), P-strong convergence with respect to an Orlicz function. Turk. Journal Math; 46 (3): 832-838.

[31] Uluçay, H., & Ünver, M., & Söylemez, D. (2023), Some Korovkin type approximation applications of power series methods. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 117(1), 24.

[32] Ünver M., & Orhan C. (2019), Statistical convergence with respect to power series methods and applications to approximation theory. Numer. Funct. Anal. Optim. 40(5), 535-547.

[33] Yıldız, S. (2021), I_2-Relative uniform convergence and Korovkin type approximation, Acta et Commentationes Universitatis Tartuensis de Mathematica 25 (2), 189-200.

[34] Yıldız, S., & Demirci, K., & Dirik, F. (2022), Korovkin theory via P_p-statistical relative modular convergence for double sequences. Rendiconti del Circolo Matematico di Palermo Series 2, 1-17.

10. ON THE SOLUTION OF AN INVERSE SOURCE PROBLEM FOR THE KINETIC EQUATION WITH A SCATTERING TERM

İsmet Gölgeleyen **Muhammed Hasdemir**

Abstract

We consider an inverse problem for a kinetic equation with a scattering term. We present existence and uniqueness theorems for the problem. Kinetic equations are used to describe time evolution of many-body systems and frequently arise in plasma physics and astrophysics. Physical interpretation of these inverse problems includes finding scattering indicatrices, forces of particle interaction and radiation sources. In this work, we also find approximate solution to the problem via a hybrid algorithm that is based on the finite difference method and trapezoidal rule. We give some computational examples to compare the exact and approximate solutions. In the both theoretical and numerical parts of our study, the main idea is to reduce the inverse problem to a direct problem of Dirichlet type for a third-order partial differential equation and then to represent the solution by using Galerkin approximation.

Keywords. Inverse problem, Kinetic equation, Finite difference method, Trapezoidal rule

ORCID: 0000-0002-3602-7219
e-mail: ismet.golgeleyen@beun.edu.tr
ORCID: 0000-0001-5901-3699
e-mail: muhammed.hasdemir@adu.edu.tr

1 Introduction

The kinetic theory constitutes a theoretical framework utilized to predict the macroscopic behavior of many-body systems. This framework examines the microscopic movements of atoms, molecules, or other particles to explane the macroscopic properties of materials, such as gases, liquids, and solids.

Kinetic equations are encountered in various fields, such as medical diagnosis and imaging [1, 2], plasma physics [3, 4], thermodynamics [5], environmental science and astrophysics [6], systems biology [7, 8], fluid mechanics and turbulence [9], and traffic flow [10, 11].

Inverse problems related to these equations involve determination of some physical parameters such as forces of particle interaction, radiation sources and scattering indicatrices. Different inverse problems regarding kinetic equations and their solvability are examined in [1, 12]. The problems involving time-independent kinetic and transport equations were solved numerically in [13, 14, 15, 16, 17]. In the time-dependent case, an approximate solution was obtained in [18] by means of symbolic computation which grounds on the method of Galerkin. The uniqueness and stability theorems on the basis of the Carleman estimate were obtained in [19, 20, 21]. On the other hand, numerical algorithms for various inverse problems are presented in [22, 23, 24].

The main goal of this paper is to solve numerically an inverse problem for a non-stationary kinetic equation with a scattering term by using the finite difference and trapezoidal methods.

2 Preliminaries

In this study, we consider the one-dimensional non-stationary kinetic equation:

$$\begin{aligned} Lu(x,v,t) &\equiv \frac{\partial u}{\partial t} + \frac{\partial H}{\partial v}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial v}\frac{\partial H}{\partial x} + I(u) = \lambda(x,v,t), \quad (2.1) \\ I(u) &= \int\limits_G \Phi\left(x,v,t,v'\right)u(x,v',t)dv', \end{aligned}$$

in the domain $Q = \left\{ (x,v,t) \middle|\ x \in D \subset \mathbb{R}, v \in G \subset \mathbb{R},\ t \in (0,T) \right\}.$

In practical applications, the unknown function u denotes the density of distribution of the particles; H is the Hamiltonian function; Φ is a scattering phase function and unknown function λ characterizes the sources.

In this paper, we investigate the following problem:

Problem 2.1. Given equation (2.1), with the right-hand side λ satisfying the differential equation

$$\widehat{L}\lambda \equiv \frac{\partial^2 \lambda}{\partial v \partial x} = 0, \tag{2.2}$$

find the unknown functions u and λ provided that

$$u|_{\partial Q} = u_0, \tag{2.3}$$

where ∂Q denotes the boundary of the domain Q.

Theorem 2.4. Let the functions $H(x, v, t) \in C^2\left(\overline{Q}\right)$ and $\Phi\left(x, v, t, v'\right) \in C^1\left(\overline{Q} \times \overline{G}\right)$ be provided. Suppose that the inequalities:

$$\frac{\partial^2 H}{\partial v^2} \geq \xi_1,\ \frac{\partial^2 H}{\partial x^2} \leq -\xi_2, \tag{2.5}$$

$$\xi_1 - 1 > 0,\ \xi_2 - \xi_3 > 0 \tag{2.6}$$

hold for all $(x, v, t) \in \overline{Q}$, where ξ_1, ξ_2 are positive numbers,

$$\xi_3 = C\left(\max_{x\in D,\, t\in(0,T)} \int_G \int_G \left(\frac{\partial \Phi}{\partial v_j}\right)^2 dv' dv\right)$$

and C is a constant stems from the inequality of Steklov. Then, Problem 2.1 has no more than one solution (u, λ), where $u \in C^2\left(\overline{Q}\right)$, $\lambda \in C^2(Q)$.

We introduce a set $\widetilde{C}_0^3(Q) = \left\{\varphi : \varphi \in C^3(Q), \varphi|_{\partial Q} = 0\right\}$. Moreover, setting $Au = \widehat{L}Lu$, we define $\Upsilon(A)$ as a set of functions u which satisfies the following two properties:

i $Au \in L_2(Q)$ for $u \in \Upsilon(A)$.

ii There is a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(Q)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$.

Although Theorem 2.4 is proven by the same way as in [1], for the completeness of the paper we provide the proof here.

Proof. Suppose that the pair of functions (u, λ) is a solution to Problem 2.1 such that $u|_{\partial Q} = 0$ and $u \in \Upsilon(A)$. From (2.1) and (2.2), it is clear that $Au = 0$. Moreover by the definition of the set $\Upsilon(A)$ there is a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(Q)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$. Observing that $u_k|_{\partial Q} = 0$, we have:

$$\frac{\partial u}{\partial x}\frac{\partial \lambda}{\partial v} = \frac{\partial}{\partial x}\left(u\frac{\partial \lambda}{\partial v}\right). \tag{2.7}$$

On the other hand, we can write:

$$\begin{aligned} &2\frac{\partial u}{\partial x}\left[\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial t}+\frac{\partial H}{\partial v}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial v}\frac{\partial H}{\partial x}+\int\limits_G \Phi\left(x,v,t,v'\right)u(x,v',t)dv\right)\right] \\ &=\frac{\partial^2 H}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u}{\partial v}\right)^2+\frac{\partial}{\partial v}\left[\frac{\partial u}{\partial x}\left(\frac{\partial H}{\partial v}\frac{\partial u}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial u}{\partial v}\right)\right] \\ &+\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\left(\frac{\partial u}{\partial v}\right)^2\right)-\frac{\partial}{\partial v}\left(\frac{\partial H}{\partial x}\frac{\partial u}{\partial x}\frac{\partial u}{\partial v}\right)+\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial v}\right)+\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) \\ &-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial t}\right)+2\frac{\partial u}{\partial x}\frac{\partial}{\partial v}\left(\int\limits_G \Phi\left(x,v,t,v'\right)u(x,v',t)dv\right)=2\frac{\partial}{\partial x}\left(u\frac{\partial \lambda}{\partial v}\right). \end{aligned} \tag{2.8}$$

By (2.7) and (2.8), we get

$$\begin{aligned} &\int\limits_Q\left(\frac{\partial^2 H}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u}{\partial v}\right)^2\right)dQ \\ &+2\frac{\partial u}{\partial x}\frac{\partial}{\partial v}\left(\int\limits_G \Phi\left(x,v,t,v'\right)u(x,v',t)dv\right)dQ \\ &=2\int\limits_Q\frac{\partial}{\partial x}\left(u\frac{\partial \lambda}{\partial v}\right)dQ=0. \end{aligned} \tag{2.9}$$

Next, we evaluate the second term on the left hand side of equality (2.9):

$$\begin{aligned} &2\int\limits_Q\left(\frac{\partial u}{\partial x}\int\limits_G\frac{\partial \Phi}{\partial v}udv'\right)dQ\geq-\int\limits_Q\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\int\limits_G\frac{\partial \Phi}{\partial v}udv'\right)^2\right)dQ \\ &\geq-\int\limits_Q\left(\frac{\partial u}{\partial x}\right)^2dQ-\int\limits_D\int\limits_G\int\limits_0^T\left(\int\limits_G\left(\frac{\partial \Phi}{\partial v}\right)^2dv'\right)\left(\int\limits_G(u)^2dv'\right)dvdxdt \\ &\geq-\int\limits_Q\left(\frac{\partial u}{\partial x}\right)^2dQ-\int\limits_D\int\limits_0^T\left(\int\limits_G\left(u(x,v',t)\right)^2dv'\right)\beta(x,t)dxdt, \end{aligned} \tag{2.10}$$

where

$$\beta(x,t)=\int\limits_G\int\limits_G\left(\frac{\partial \Phi}{\partial v}\right)^2dv'dv. \tag{2.11}$$

In (2.10), we use the inequality $2ab\geq-a^2-b^2$, Cauchy-Schwartz inequality and Fubini Theorem, respectively. By using the Steklov inequality, we have

$$-\int\limits_D\int\limits_0^T\left(\int\limits_G(u)^2dv'\right)\beta(x,t)dxdt\geq-C\left(\max_{x\in D,\,t\in(0,T)}\beta(x,t)\right)\int\limits_Q\left(\frac{\partial u}{\partial v}\right)^2dQ$$

and

$$2\int_Q \left(\frac{\partial u}{\partial x}\int_G \frac{\partial \Phi}{\partial v} u dv'\right) dQ \geq -\int_Q \left(\frac{\partial u}{\partial x}\right)^2 dQ - \alpha_3 \int_Q \left(\frac{\partial u}{\partial v}\right)^2 dQ, \tag{2.12}$$

where $\xi_3 = C\left(\max_{x\in D,\, t\in(0,T)} \beta(x,t)\right)$. Thus, using (2.12) and condition (2.5) in equality (2.9) we have

$$\begin{aligned}
&\int_Q \left(\frac{\partial^2 H}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial^2 H}{\partial x^2}\left(\frac{\partial u}{\partial v}\right)^2\right) dQ \\
&+2\int_Q \left(\frac{\partial u}{\partial x}\int_G \frac{\partial}{\partial v}\left(\Phi\left(x,v,t,v'\right) u(x,v',t)dv\right)\right) dQ \\
&\geq (\xi_1 - 1)\int_Q \left(\frac{\partial u}{\partial x}\right)^2 dQ + (\xi_2 - \xi_3)\int_Q \left(\frac{\partial u}{\partial v}\right)^2 dQ \leq 0.
\end{aligned}$$

Therefore, by (2.6) we have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} = 0$. Taking into account the definition of $\Upsilon(A)$ we can write $\int_Q |\nabla u|^2 dQ \leq 0$, which implies that $u = 0$ in Q because of the boundary condition $u|_{\partial Q} = 0$. Then, by Eq. (2.1), we obtain $\lambda = 0$. Thus the proof of Theorem 2.4 is completed. □

In order to prove the existence of the solution of Problem 2.1, we shall use the Galerkin method and therefore we deal with the following problem:

Problem 2.2. Find the pair of functions (u, λ) from the following relations:

$$Lu = \lambda(x,v,t) + F, \tag{2.13}$$

$$u|_{\partial Q} = 0 \tag{2.14}$$

provided that $F \in H_2(Q)$.

Theorem 2.15. Based on the hypothesis presented in Theorem 2.4, there exists a solution (u, λ) to Problem 2.2 in $H_1(Q)\times L_2(Q)$.

Theorem 2.15 can be proven by the same way as in[1], (Theorem 1.1.2).

3 Numerical Solution of Problem 1

In this section, we present a method to solve Problem 1 numerically. For this aim, the operator $\widehat{L}$ is applied to the both sides of equation (2.13), then we have:

$$u_{txv} + u_{xvx}H_v - u_{vvx}H_x + u_{xx}H_{vv} - u_{vv}H_{xx} + u_xH_{vvx} - u_vH_{xvx} + \int\limits_G \Phi_{xv}udv' + \int\limits_G \Phi_v u_x dv' = \widehat{L}F. \tag{3.1}$$

Employing the finite difference formulas and trapezoidal rule in (3.1), we obtain

$$\begin{aligned}
&(-a_2 + a_1)\,\tilde{u}^n_{j-1,k-1} + (2a_2 - a_3 + a_5)\,\tilde{u}^n_{j,k-1} + (-a_1 - a_2)\,\tilde{u}^n_{j+1,k-1}\\
&+ (-2a_1 + a_4 - a_6)\,\tilde{u}^n_{j-1,k} + (-2a_4 + 2a_3)\,\tilde{u}^n_{j,k} + (a_2 - a_1)\,\tilde{u}^n_{j+1,k+1}\\
&+ (-2a_1 + a_4 - a_6)\,\tilde{u}^n_{j-1,k} + (-2a_4 + 2a_3)\,\tilde{u}^n_{j,k} + (a_2 - a_1)\,\tilde{u}^n_{j+1,k+1}\\
&+(a_7)(\tilde{u}^{n+1}_{j+1,k+1} - \tilde{u}^{n+1}_{j-1,k+1} - \tilde{u}^{n+1}_{j+1,k-1} + \tilde{u}^{n+1}_{j-1,k-1} - \tilde{u}^{n-1}_{j+1,k+1}\\
&+\tilde{u}^{n-1}_{j-1,k+1} + \tilde{u}^{n-1}_{j+1,k-1} - \tilde{u}^{n-1}_{j-1,k-1}) + a_{10} + a_{11} + \sum_{k=1}^{K+1} a_8\left(\tilde{u}^n_{j,k}\right)\\
&+ \sum_{k=1}^{K+1} a_9(\tilde{u}^n_{j+1,k} - \tilde{u}^n_{j-1,k}) = F^n_{j,k}\ ,\\
&j = \overline{1, J}, k = \overline{1, K}, n = \overline{1, N}\ ,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
a_1 &= \frac{(H_x)^n_{j,k}}{2\,(\Delta x)\,(\Delta v)^2},\ a_2 = \frac{(H_v)^n_{j,k}}{2\,(\Delta x)^2\,(\Delta v)},\ a_3 = \frac{(H_{xx})^n_{j,k}}{(\Delta v)^2},\\
a_4 &= \frac{(H_{vv})^n_{j,k}}{(\Delta x)^2},\ a_5 = \frac{(H_{vxx})^n_{j,k}}{2\Delta v},\ a_6 = \frac{(H_{vvx})^n_{j,k}}{2\Delta x},\\
a_7 &= \frac{1}{8\,(\Delta x)\,(\Delta v)\,(\Delta t)},\ a_8 = \frac{\Phi_{xv}}{\Delta v},\ a_9 = \frac{\Phi_v}{2\Delta v},\\
a_{10} &= \frac{\Delta v}{2}\left(\Phi_{xv}(x,v,t,c) \times u(x,c,t) + \Phi_{xv}(x,v,t,d) \times u(x,d,t)\right),\\
a_{11} &= \frac{\Delta v}{2}\left(\Phi_v(x,v,t,c) \times u_x(x,c,t) + \Phi_v(x,v,t,d) \times u_x(x,d,t)\right).
\end{aligned}$$

By (2.3), we have discrete version of boundary condition

$$\begin{aligned}
&\tilde{u}^n_{0,k} = u(a, v_k, t_n), \tilde{u}^n_{J+1,k} = u(b, v_k, t_n), \tilde{u}^n_{j,0} = u(x_j, c, t_n),\\
&\tilde{u}^n_{j,K+1} = u(x_j, d, t_n), \tilde{u}^0_{j,k} = u(x_j, v_k, 0), \tilde{u}^{N+1}_{j,k} = u(x_j, v_k, T),\\
&j = \overline{0, J+1}, k = \overline{0, K+1}, n = \overline{0, N+1},
\end{aligned} \tag{3.3}$$

where we define the step sizes $\Delta x = \frac{b-a}{J+1}$, $\Delta v = \frac{d-c}{K+1}$ and $\Delta t = \frac{T}{N+1}$ for positive integers J, K, and N in the directions x, v, and t, respectively.

In (3.2) and (3.3), the notations $\tilde{u}^n_{j,k}$, $h^n_{j,k}$, $\varphi^n_{j,k}$ indicate the finite difference approximation to the functions $u(x_j, v_k, t_n) = u(a+j\Delta x, c+k\Delta v, n\Delta t)$, $H(x_j, v_k, t_n) = H(a + j\Delta x,\ c + k\Delta v,\ n\Delta t)$ and $\Phi(x_j, v_k, t_n, v'_k) = \Phi(a + j\Delta x, c + k\Delta v,\ n\Delta t, c + k\Delta v)$, respectively.

We can write (3.2) and (3.3) in the following matrix form:

$$\tilde{A}\widetilde{\mathbf{u}} = \mathfrak{F}. \tag{3.4}$$

Here the block tridiagonal banded matrix $\tilde{A}$ is given by

$$\tilde{A} = \begin{bmatrix} \mathcal{P}^{(1)} & \mathcal{R}^{(1)} & 0 & \cdots & 0 \\ \mathcal{S}^{(2)} & \mathcal{P}^{(2)} & \mathcal{R}^{(2)} & \ddots & \vdots \\ 0 & \mathcal{S}^{(3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \mathcal{R}^{(J-1)} \\ 0c\ldots & 0 & \mathcal{S}^{(J)} & \mathcal{P}^{(J)} & \end{bmatrix}_{JKN\times JKN},$$

where

$$\mathcal{P}^{(j)} = \begin{bmatrix} P_1^{(j,1)} & P_2^{(j,1)} & P_4^{(j,1)} & \cdots & P_4^{(j,1)} & P_4^{(j,1)} \\ P_3^{(j,2)} & P_1^{(j,2)} & P_2^{(j,2)} & P_4^{(j,2)} & \ddots & \vdots \\ P_4^{(j,3)} & P_3^{(j,3)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & P_4^{(j,K-2)} \\ P_4^{(j,K-1)} & \cdots & P_4^{(j,K-1)} & \ddots & \ddots & P_2^{(j,K-1)} \\ P_4^{(j,K)} & \cdots & \cdots & P_4^{(j,K)} & P_3^{(j,K)} & P_1^{(j,K)} \end{bmatrix}_{KN\times KN},$$

$j = \overline{1, J}$,

$$\mathcal{R}^{(j)} = \begin{bmatrix} R_1^{(j,1)} & R_2^{(j,1)} & R_4^{(j,1)} & \cdots & R_4^{(j,1)} & R_4^{(j,1)} \\ R_3^{(j,2)} & R_1^{(j,2)} & R_2^{(j,2)} & R_4^{(j,2)} & \ddots & \vdots \\ R_4^{(j,3)} & R_3^{(j,3)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & R_4^{(j,K-2)} \\ R_4^{(j,K-1)} & \cdots & R_4^{(j,K-1)} & \ddots & \ddots & R_2^{(j,K-1)} \\ R_4^{(j,K)} & \cdots & \cdots & R_4^{(j,K)} & R_3^{(j,K)} & R_1^{(j,K)} \end{bmatrix}_{KN\times KN},$$

$j = \overline{1, J-1}$,

$$\mathcal{S}^{(j)}=\begin{bmatrix} S_1^{(j,1)} & S_2^{(j,1)} & S_4^{(j,1)} & \cdots & S_4^{(j,1)} & S_4^{(j,1)} \\ S_3^{(j,2)} & S_1^{(j,2)} & S_2^{(j,2)} & S_4^{(j,2)} & \ddots & \vdots \\ S_4^{(j,3)} & S_3^{(j,3)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & S_4^{(j,K-2)} \\ S_4^{(j,K-1)} & \cdots & S_4^{(j,K-1)} & \ddots & \ddots & S_2^{(j,K-1)} \\ S_4^{(j,K)} & \cdots & \cdots & S_4^{(j,K)} & S_3^{(j,K)} & S_1^{(j,K)} \end{bmatrix}_{KN\times KN},$$

$j=\overline{2,J}$,

and

$$P_m^{(j,k)}=\begin{bmatrix} f_m(j,k) & 0 & \cdots & 0 \\ 0 & f_m(j,k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f_m(j,k) \end{bmatrix}_{N\times N},$$

$$R_m^{(j,k)}=\begin{bmatrix} g_m(j,k) & -\mu a_7 & 0 & \cdots & 0 \\ \mu a_7 & g_m(j,k) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\mu a_7 \\ 0 & \cdots & 0 & \mu a_7 & g_m(j,k) \end{bmatrix}_{N\times N},$$

$$S_m^{(j,k)}=\begin{bmatrix} h_m(j,k) & \mu a_7 & 0 & \cdots & 0 \\ -\mu a_7 & h_m(j,k) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \mu a_7 \\ 0 & \cdots & 0 & -\mu a_7 & h_m(j,k) \end{bmatrix}_{N\times N},$$

$$\mu=\frac{(-1)^{(m+1)}(m-1)}{(m-1)!}, m=1,2,3\,,\ \ k=\overline{1,K}$$

$$P_4^{(j,k)} = \begin{bmatrix} a_8(j,k) & 0 & \cdots & 0 \\ 0 & a_8(j,k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_8(j,k) \end{bmatrix}_{N\times N},$$

$$R_4^{(j,k)} = \begin{bmatrix} a_9(j,k) & 0 & \cdots & 0 \\ 0 & a_9(j,k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_9(j,k) \end{bmatrix}_{N\times N},$$

$$S_4^{(j,k)} = \begin{bmatrix} -a_9(j,k) & 0 & \cdots & 0 \\ 0 & -a_9(j,k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -a_9(j,k) \end{bmatrix}_{N\times N},$$

and $f_1(j,k) = -2a_4+2a_3+a_8$, $f_2(j,k) = -2a_2-a_3-a_5+a_8$, $f_3(j,k) = 2a_2-a_3+a_5+a_8$, $g_1(j,k) = -2a_1+a_4-a_6-a_9$, $g_2(j,k) = a_1+a_2-a_9$, $g_3(j,k) = a_1 - a_2 - a_9$, $h_1(j,k) = 2a_1 + a_4 + a_6 + a_9$, $h_2(j,k) = -a_1 + a_2 + a_9$, $h_3(j,k) = -a_1 - a_2 + a_9$. By solving (3.4), we obtain the approximate solution vector

$$\widetilde{\mathbf{u}} = \left[\tilde{u}_{1,1}^1, \tilde{u}_{1,1,}^2, ..., \tilde{u}_{1,1}^N, \tilde{u}_{1,2}^1, \tilde{u}_{1,2}^2, ..., \tilde{u}_{1,2}^N, ..., \tilde{u}_{1,K}^1, \tilde{u}_{1,K}^2, ..., \tilde{u}_{1,K}^N, ..., \tilde{u}_{J,K}^N\right]^T$$

of Problem 2.1 at JKN mesh points of Q.

Numerical solution for λ can be obtained using the approximate values $\tilde{u}_{j,k}^n$ from the difference equation

$$\begin{gathered} \frac{\tilde{u}_{j,k}^{n+1} - \tilde{u}_{j,k}^{n-1}}{2\Delta t} + a_2 \frac{\tilde{u}_{j+1,k}^n - \tilde{u}_{j-1,k}^n}{2\Delta x} - a_1 \frac{\tilde{u}_{j,k+1}^n + \tilde{u}_{j,k-1}^n}{2\Delta v} \\ +a_{12} + (\Delta v) \sum_{k=1}^{K+1} \left(\Phi_{j,k}^n\right)\left(\tilde{u}_{j,k}^n\right) = \tilde{\lambda}_{j,k}^n, \end{gathered} \tag{3.5}$$

which is a discrete form of (2.13) for $j = \overline{1,J}$; $k = \overline{1,K}$; $n = \overline{1,N}$. In (3.5),

$$a_{12} = \frac{\Delta v}{2}\left(\Phi(x,v,t,c) \times u(x,c,t) + \Phi(x,v,t,d) \times u(x,d,t)\right) \tag{3.6}$$

and $\tilde{\lambda}_{j,k}^n$ is the approximation to the unknown source function $\lambda(x_j, v_k, t_n) = \lambda(a + j\Delta x,\ c + k\Delta v,\ n\Delta t)$.

4 Numerical Experiments

The solution algorithm introduced above has been implemented and analyzed on various inverse source problems for kinetic equations. Some of the examples are demonstrated below. The figures we provided shows the relation between analytical and numerical solutions.

Example 4.1. Determine (u, λ) in the domain $Q = (2,3) \times (0,2) \times (4,7)$ from relations (2.13) and (2.14) provided that

$$\Phi\left(x,v,t,s\right) = e^{v-s},\ H\left(x,v,t\right) = \ln x + v$$

$$
\begin{aligned}
F \quad = \quad & \frac{1}{x^2(t-3)}\left(-98v^3x^3+49v^4x^3+28tv^3x^3-14tv^4x^3+t^2v^4x^3\right.\\
&-2t^2v^3x^3+448tv^3x-224tv^4x-1568v^3x+784v^4x-32t^2v^3x\\
&\left.+16t^2v^4x+1176v^3-588v^4+24t^2v^3-12t^2v^4-336tv^3+168tv^4\right)\\
&-\frac{\ln(t-3)}{x^3}\left(-2058v^2x^2-224v^4x^2-353.46x^2e^v-42t^2v^2x^2\right.\\
&+28t^2v^3x^2-7.2135t^2x^2e^v+588tv^2x^2-456tv^3x^2+32tv^4x^2\\
&+100.98tx^2e^v+4704v^2x-1904v^3x-616v^4x+265.09xe^v\\
&+96t^2v^2x-32t^2v^3x-16t^2v^4x+5.41t^2xe^v-1344tv^2x+496tv^3x\\
&+200tv^4x-75.74txe^v+1820v^3-3528v^2+1176v^4-72t^2v^2\\
&\left.+24t^2v^4+1008tv^2-336tv^4\right)+x\ln(t-3)\left(-22.0915e^v+28v^3\right.\\
&\left.-14v^4-0.45t^2e^v-4tv^3+2tv^4+6.31te^v\right).
\end{aligned}
$$

Here, the analytical solution of the problem is

$$
\begin{aligned}
u\,(x,v,t) \quad &= \quad v^3(x-3)(v-2)\left(\frac{2}{x}-1\right)^2(t-7)^2\ln(t-3),\\
\lambda\,(x,v,t) \quad &= \quad \left(\frac{1}{(t-3)}\right)(14t^2v^3-196tv^3+686v^3-7t^2v^4+98tv^4-343v^4)\\
&\quad +\ln(t-3)(294v^2-490v^3+147v^4+6t^2v^2-6t^2v^3+t^2v^4\\
&\quad -84tv^2+112tv^3)+28tv^4+e^v(-4116-84t^2+1176t\\
&\quad +31556e^{(-2)}-9016te^{(-2)}+644t^2v).
\end{aligned}
$$

The exact and approximate solution for Example 4.1 are provided in the following figures:

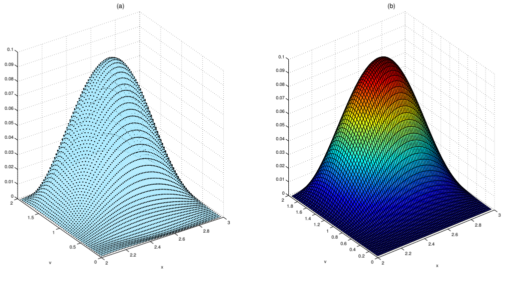


Figure 14: *(a) Numerical solution, (b) Exact solution for u at $t=5$*

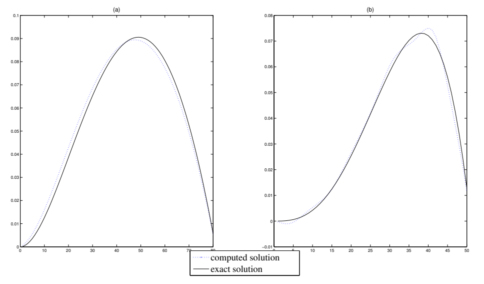


Figure 15: *Exact and approximate solutions for* u*: (a) varying* x*;(b) varying* v

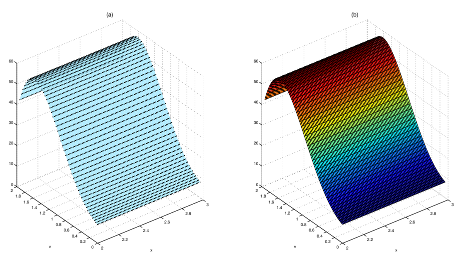


Figure 16: *(a) Numerical solution and (b) Exact solution for* λ *at* $t = 5$

Example 4.2. Let us consider the problem of finding (u, λ) defined in $Q = (-5, 3) \times (0, 3) \times (0, 7)$ that satisfies relations (2.13) and (2.14) with the provided functions

$$\Phi(x, v, t, s) = s + t, \ H(x, v, t) = -x^2$$

and

$$\begin{aligned} F &= 408tvx - 14v^2x + 21vx^2 + 4t^2vx^3 + 4tv^2x - 62tvx^2 - 7v^2x^2 \\ &\quad -28tvx^3 + 8t^2vx^2 - 60t^2vx + 2tv^2x^2 + 42vx. \end{aligned}$$

The analytical solution of the problem:

$$\begin{aligned}\lambda(x,v,t) &= -9t^3x - 6t^2x^3 - 30tv^2 + 90tv + 42tx^3 + 105v^2 \\ &\quad -315v + (51t^2x^2 - 1485t^2 + 525tx^2 - 2835t)/4 \\ &\quad -(9t^3x^2 - 135t^3 - 279t^2x + 1071tx)/2, \\ u(x,v,t) &= (x^2+2x-15)(v^2-3v)(t^2-7t).\end{aligned}$$

By the proposed method, the following numerical results are obtained for (u, λ):

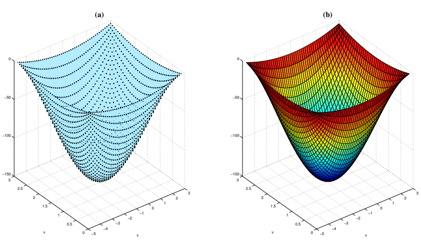


Figure 17: *(a) Numerical solution and (b) Exact solution for u at $t = 4$*

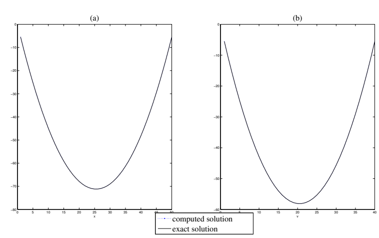


Figure 18: *; Exact and approximate solutions for u: (a) varying x; (b) varying v*

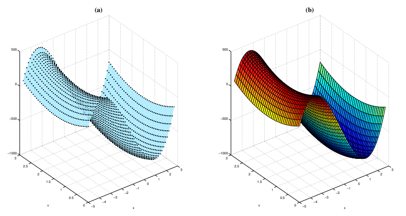


Figure 19: *(a) Numerical solution, (b) Exact solution for λ at $t = 4$*

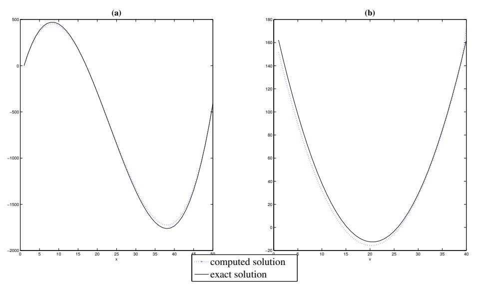


Figure 20: *Approximate and exact solutions for λ: (a) varying x; (b) varying v*

Example 4.3. Find u and λ defined in $Q = (1,4) \times (0,1) \times (0,5)$ that satisfies (2.13) and (2.14) with the provided functions

$$\Phi\left(x,v,t,s\right) = xv(s+t), H\left(x,v,t\right) = -x^2 \tag{4.4}$$

and

$$\begin{aligned} F &= -(v\sin(\pi x)(2^x-2)(v-19.58x+4.75tx+3^t\left(1.09t\right)-0.16t^2x \\ &\quad +3^t\left(4.49v+19.58x-4.49-1.09tv-4.75tx+0.16t^2x-1\right). \end{aligned}$$

Here, we know that the solution of this problem is

$$\begin{aligned} u\left(x,v,t\right) &= v(v-1)(t-5)(3^t-1)(2^x-2)\sin(x\pi), \\ \lambda\left(x,v,t\right) &= x\sin(\pi x)\left(20-4t+3^t\left(4t-20\right)+\left(2^x+2^x3^t+\right)\left(2t-10\right)\right). \end{aligned}$$

In the following graphs, we present a comparisons of the exact and numerical solution for (u, λ):

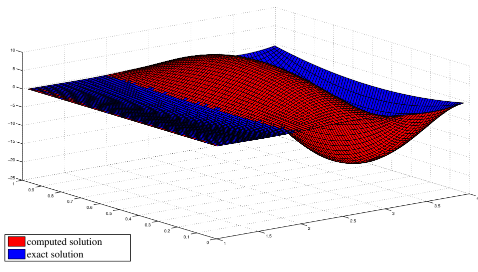


Figure 21: *Approximate and exact solutions for* u *at* $t = 3$

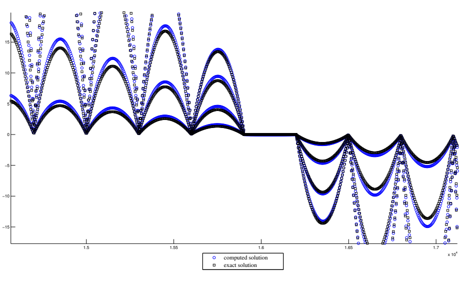


Figure 22: *Numerical and exact solutions for* u *at all points*

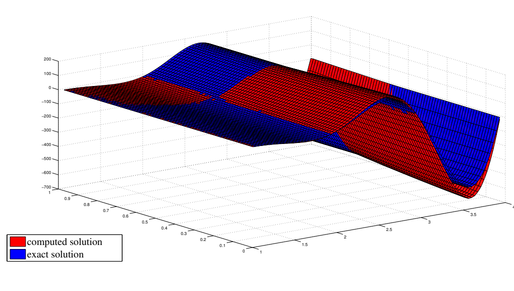


Figure 23: *Approximate and exact solutions for* λ *at* $t = 3$

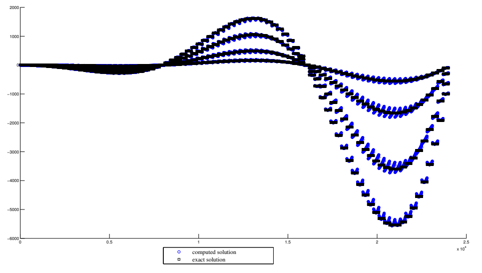


Figure 24: *(a) Calculated solution, (b) exact solution for λ at all points*

5 Conclusion

As a result, numerical experiments demonstrated that the proposed method offers effective and reliable numerical solutions for inverse problems involving the general kinetic equation. Notably, the method employed to investigate the existence and uniqueness of solution to Problem 2.1 paves a way for creating a computational method to get an approximate solution to the problem. Namely, applying the operator $\hat{L}$, Problem 2.1 is replaced with a Dirichlet Problem for a third order partial differential equation. Then, the approximate values are obtained by a combination of finite difference method and trapezoidal rule. This study provides the opportunity to study on different problems because it is applicable to both forward problems involving third-order partial differential equations and related inverse problems.

References

[1] Amirov A. K. (2001) Integral geometry and inverse problems for kinetic equations, VSP, Utrecht, The Netherlands.

[2] Mueller, J. L., & Siltanen, S. (Eds.). (2012). Linear and nonlinear inverse problems with practical applications. Society for Industrial and Applied Mathematics.

[3] Mouhot C., & Villani C. (2011). On Landau damping. Acta Mathematica 207, 29-201.

[4] Chavanis, P. H. (2012). Kinetic theory of spatially homogeneous systems with long-range interactions: I. General results. The European Physical Journal Plus, 127, 1-23.

[5] Bikkin, H., & Lyapilin, I. I. (2021). Non-equilibrium thermodynamics and physical kinetics. Walter de Gruyter GmbH & Co KG.

[6] Vereshchagin, G. V., & Aksenov, A. G. (2017). Relativistic kinetic theory: with applications in astrophysics and cosmology. Cambridge University Press.

[7] Fiasconaro, A., & Spagnolo, B., & Ochab-Marcinek, A., & Gudowska-Nowak, E. (2006). Co-occurrence of resonant activation and noise-enhanced stability in a model of cancer growth in the presence of immune response. Physical Review E—Statistical, Nonlinear, and Soft Matter Physics, 74(4), 041904.

[8] Bellomo N., & Bellouquid A. , & Delitala M. (2004). Mathematical topics on the modelling complex multicellular systems and tumor immune cells competition. Mathematical Models and Methods in Applied Sciences 14, 1683-1733.

[9] Chen, H., & Kandasamy, S., & Orszag, S., & Shock, R., & Succi, S., & Yakhot, V. (2003). Extended Boltzmann kinetic equation for turbulent flows. Science, 301(5633), 633-636.

[10] Klar, A., & Wegener, R. (2000). Kinetic traffic flow models. In Modeling in Applied Sciences: A Kinetic Theory Approach (pp. 263-316). Boston, MA: Birkhäuser Boston.

[11] Delitala, M. (2003). Nonlinear models of vehicular traffic flow–new frameworks of the mathematical kinetic theory. Comptes Rendus Mecanique, 331(12), 817-822.

[12] Anikonov, Y. E., & Amirov, A. K. (1983). A uniqueness theorem for the solution of an inverse problem for a kinetic equation. In Doklady Akademii Nauk (Vol. 272, No. 6, pp. 1292-1293). Russian Academy of Sciences.

[13] Gölgeleyen, İ. (2013). An inverse problem for a generalized transport equation in polar coordinates and numerical applications. Inverse problems, 29(9), 095006.

[14] Amirov, A., & Ustaoglu, Z., & Heydarov, B. (2011). Solvability of a two dimensional coefficient inverse problem for transport equation and a

numerical method. Transport theory and statistical physics, 40(1), 1-22.

[15] Amirov, A., & Golgeleyen, F., & Rahmanova, A. (2009). An inverse problem for the general kinetic equation and a numerical method. Computer Modeling in Engineering and Sciences (CMES), 43(2), 131.

[16] Gölgeleyen F., & Hasdemir M. (2019). Numerical solution of an inverse problem for the Liouville Equation. TWMS Journal of Applied and Engineering Mathematics 9, 909-920.

[17] Gölgeleyen I., & Hasdemir M. (2022). A solution algorithm for an inverse source problem for the kinetic equation. International Journal of Modern Physics C 33, 1-18.

[18] Yıldız M., & Heydarov B., & Gölgeleyen I. (2010). Approximate solution of an inverse problem for a non-stationary general kinetic equation. Computer Modeling in Engineering and Sciences 62, 255-264.

[19] Lai R. Y., & Li Q. (2020). Parameter reconstruction for general transport equation. SIAM Journal on Mathematical Analysis 52, 2734-2758.

[20] Klibanov M. V., & Li J. (2021). Inverse problems and Carleman estimates. De Gruyter.

[21] Kaytmaz Ö. (2024). The problem of determining source term in a kinetic equation in an unbounded domain. AIMS Mathematics 9, 9184-9194.

[22] Karchevsky A. L., & Romanov V. G. (2019). A numerical method of determining permittivity from the modulus of the electric intensity vector of an electromagnetic field. Journal of Applied and Industrial Mathematics 13, 436-446.

[23] Kabanikhin S. I., & Sabelfeld K. K., & Novikov N. S., & Shishlenin M. A. (2015). Numerical solution of an inverse problem of coefficient recovering for a wave equation by a stochastic projection methods. Monte Carlo Methods and Applications 21, 189-203.

[24] Kanca F., & Ismailov M. I. (2012). The inverse problem of finding the time-dependent diffusion coefficient of the heat equation from integral overdetermination data. Inverse Problems in Science and Engineering 20, 463-476.

11. TWO WEIGHTED INEQUALITIES FOR RIESZ POTENTIALS IN GENERALIZED WEIGHTED MORREY SPACE

Fatma GELERİ **Canay AYKOL**

Abstract

In this paper, the definition of Morrey space $L_{p,\lambda}$ and generalized weighted Morrey space $\mathcal{M}_{\omega}^{p,\varphi}$, where ω weights belong to the Fefferman-Pong class, will be given. Then, a theorem will be given with the definition of the Hardy operator. Afterwards, two weighted inequalities will be obtained for Riesz potential I_α. Finally, with the help of the obtained inequality, it will be proven that the boundedness of Riesz potential I_α from the spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$.

Keywords. Riesz potential, Morrey space, Generalized weighted Morrey space, Fefferman-Pong class

ORCID: 0009-0007-7016-0367
e-mail: fgeleri@ankara.edu.tr
ORCID: 0000-0002-2854-6369
e-mail: canay.aykol@science.ankara.edu.tr

1 Introduction

Let $0 < \alpha < n$ and $f \in L_1^{loc}(\mathbb{R}^n)$. Then the Riesz potential operator I_α is defined by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

I_α Riesz potential is one of the basic tools of harmonic analysis used in the solution of partial differential equations. Since $0 < \alpha < n$, Riesz potential operator I_α is a weak singular operator. The boundedness of this operator in various function spaces have been studied by many mathematicians. Some of these spaces are Morrey space $L_{p,\lambda}$, generalized Morrey space $\mathcal{M}^{p,\varphi}$, generalized weighted Morrey space $\mathcal{M}^{p,\varphi}_{\omega}$ and global generalized weighted Morrey space $G\mathcal{M}_{p,\theta,\varphi,\omega}$. The boundedness of the Riesz potential in Morrey spaces was obtained by Adams in 1975 ([1]). In [5], Guliyev obtained the boundedness of Riesz potential, maximal and singular integral operators in generalized Morrey spaces. In 2012, Guliyev generalized both the generalized Morrey spaces and the weighted Morrey spaces and defined the generalized weighted Morrey spaces (see [6]). The boundedness of the Riesz potential, singular and maximal operators in weighted Lebesgue spaces mentioned in the definition of the generalized weighted Morrey space was obtained by Muckenhoupt and Wheeden in [9] and also by Coifman and Fefferman in [4]. The boundedness of the Riesz potential in the generalized weighted Morrey spaces has been proven by Aykol, Hasanov and Safarov in [2]. The boundedness of the Riesz potential in global generalized weighted Morrey spaces with the Fefferman-Pong weight class has been proved by Aykol, Geleri, Hasanov and Safarov in [3].

In this paper, we give the definition of generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi}_{\omega}$, where ω weights belong to the Fefferman-Pong class. Then, we obtain a two weighted inequality for Riesz potential I_α. We prove the boundedness of Riesz potential I_α from the spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$ with the help of the obtained inequality.

Definition 1.1. Let $0 < \lambda < n$, $p \in [1, \infty)$ and $f \in L^{loc}_p(\mathbb{R}^n)$. Then the Morrey space $L_{p,\lambda}$ is defined by

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_{p(B(x,r))}}$$
$$= \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \left(\int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Morrey spaces were introduced by C.B. Morrey [8] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Then, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients and potential theory.

Definition 1.2. Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and ω be a weight. We denote by the generalized weighted

Morrey space $\mathcal{M}_{\omega}^{p,\varphi}(\mathbb{R}^n)$, the space of all functions $f \in L_{p,\omega}^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{\mathcal{M}_{\omega}^{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^{\frac{n}{pt'}}\varphi(r)\|\omega\|_{L_{pt}(B(x,r))}} \|f\|_{L_{p,\omega}(B(x,r))},$$

where $f \in L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \|f_{\chi(B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{(B(x,r)} |f(y)|^p \omega(y)d(y)\right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}_{\omega}^{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,\omega}^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{W\mathcal{M}_{\omega}^{p,\varphi}((\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^{\frac{n}{pt'}}\varphi(r)\|\omega\|_{L_{pt}(B(x,r))}} \|f\|_{WL_{p,\omega}(B(x,r))},$$

where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_{p,\omega}((B(x,r))} &\equiv \|f_{\chi_{B(x,r)}}\|_{WL_{p,\omega}(\mathbb{R}^n)} \\ &= \sup_{t>0} t \left(\int_{y\in(B(x,r)):|f(y)|>t} |f(y)|^p \omega(y)d(y)\right)^{\frac{1}{p}}.\end{aligned}$$

Definition 1.3. The weight functions (ω_1, ω_2) belong to the class $A_{p,q}(\mathbb{R}^n)$ for $p, q \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, if the following statement

$$\sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}-1} \left(\int_{B(x,r)} \omega_2^q(y)dy\right)^{\frac{1}{q}} \left(\int_{B(x,r)} \omega_1^{-p'}(y)dy\right)^{\frac{1}{p'}} < \infty$$

is satisfied.

In this section, we give the definition of the Fefferman-Pong weight class that we use in the proofs.

Definition 1.4. [10] The weight functions (ω_1, ω_2) belong to the class $F_{p,q}(\mathbb{R}^n)$ for $1 < p, q, t < \infty$, if the following statement

$$\sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}-\frac{1}{qt}-\frac{1}{p't}} \|\omega_2\|_{L_{qt}(B(x,r))} \left\|\omega_1^{-1}\right\|_{L_{p't}(B(x,r))} < \infty$$

is satisfied.

Now we give the definition of the Hardy operator.

Definition 1.5. Let $0 < r < \infty$. Then the Hardy operator H_w^* is defined by

$$H_w^* g(r) := \int_r^\infty g(s)w(s)ds,$$

where w is a weight.

Theorem 1.6. [7] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(r)$ be bounded outside a neighborhood of the origin. The inequality

$$ess \sup_{r>0} v_2(r) H_w^* g(r) \leq C ess \sup_{r>0} v_1(r) g(r)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(\tau)d\tau}{\operatorname{ess\,sup}_{\tau<s<\infty} v_1(s)} < \infty.$$

2 Two Weighted Inequalities for Riesz Potential in Generalized Weighted Morrey Spaces

In this section we prove the boundedness of the Riesz potential operator I_α from the spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\mathbb{R}^n)$. First, we give a theorem expressing two weighted boundedness of the Riesz potential I_α from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{q,\omega_2}(\mathbb{R}^n)$.

Theorem 2.1. [10] Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $(\omega_1, \omega_2) \in F_{p,q}(\mathbb{R}^n)$. Then the operator I_α is bounded from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{q,\omega_2}(\mathbb{R}^n)$.

Theorem 2.2. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $(\omega_1, \omega_2) \in F_{p,q}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for an arbitrary $f \in L_{p,\omega_1}(\mathbb{R}^n)$ the inequality

$$\|I_\alpha f\|_{L_{q,\omega_2}(B(x,r))} \leq C r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s} \tag{2.3}$$

is hold.

Proof. We represent f as

$$f_1(y) = f(y)\chi_{B(x,2r)}(y) \quad and \quad f_2(y) = f(y)\chi_{^cB(x,2r)}(y)$$

and have

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

By Theorem 2.1, we obtain

$$\|I_\alpha f_1\|_{L_{q,\omega_2}(B(x,r))} \leq \|I_\alpha f_1\|_{L_{q,\omega_2}(\mathbb{R}^n)} \leq C\,\|f_1\|_{L_{p,\omega_1}(\mathbb{R}^n)} = C\,\|f\|_{L_{p,\omega_1}(B(x,2r))}\,.$$

Then

$$\|I_\alpha f_1\|_{L_{q,\omega_2}(B(x,r))} \leq C\,\|f\|_{L_{p,\omega_1}(B(x,2r))}\,, \tag{2.4}$$

where the constant C is independent of f. With the help of inequality (2.4) we get

$$\|I_\alpha f_1\|_{L_{q,\omega_2}(B(x,r))} \leq C r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s}. \tag{2.5}$$

When $|x-z| \leq r$ and $|z-y| \geq 2r$, we have

$$\frac{1}{2}\,|z-y| \leq |x-y| \leq \frac{3}{2}\,|z-y|\,.$$

Therefore we get

$$|I_\alpha f_2(z)| \leq \int\limits_{{}^cB(x,2r)} |z-y|^{\alpha-n}\,|f(y)|\,dy \leq C \int\limits_{{}^cB(x,2r)} |x-y|^{\alpha-n}\,|f(y)|\,dy.$$

Then we obtain

$$\begin{aligned}
\int\limits_{{}^cB(x,2r)} |x-y|^{\alpha-n}\,|f(y)|\,dy &= C \int\limits_{{}^cB(x,2r)} |f(y)| \left(\int\limits_{|x-y|}^{\infty} s^{\alpha-n-1} ds \right) dy \\
&= C \int\limits_{2r}^{\infty} s^{\alpha-n-1} \left(\int\limits_{\{y\in\mathbb{R}^n : 2r \leq |x-y| \leq s\}} |f(y)|\,dy \right) ds \\
&\leq C \int\limits_{r}^{\infty} s^{\alpha-n-1}\, \|f\|_{L_{p,\omega_1}(B(x,s))} \left\|\omega_1^{-1}\right\|_{L_{p'}(B(x,s))} ds \\
&\leq C \int\limits_{r}^{\infty} s^{\alpha-n+\frac{n}{p't'}-1}\, \|f\|_{L_{p,\omega_1}(B(x,s))} \left\|\omega_1^{-1}\right\|_{L_{p'}t(B(x,s))} ds \\
&\leq C \int\limits_{r}^{\infty} s^{\alpha-n+\frac{n}{p't'}-1-\alpha+\frac{n}{p}-\frac{n}{q}+\frac{n}{p't}+\frac{n}{qt}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} ds \\
&= C \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|I_\alpha f_2\|_{L_{q,\omega_2}(B(x,r))} &\leq C\|\omega_2\|_{L_q(B(x,r))} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s} \\
&\leq C r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s}.
\end{aligned}$$

Then we have

$$\|I_\alpha f_2\|_{L_{q,\omega_2}(B(x,r))} \le C r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s}. \tag{2.6}$$

Therefore we get which together with (2.5) and (2.6) yields (2.3). □

In the following theorem, we prove the boundedness of the Riesz potential operator I_α from the spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$. We find conditions on the functions $\varphi_1(r)$ and $\varphi_2(r)$ for the boundedness of I_α from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Theorem 2.7. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} = \frac{\alpha}{n} + \frac{1}{q}$ and $(\omega_1, \omega_2) \in F_{p,q}(\mathbb{R}^n)$. The functions $\varphi_1(r)$ and $\varphi_2(r)$ fulfil the condition

$$\int_r^\infty s^{-\frac{n}{qt'}} \frac{\operatorname*{ess\,inf}_{s<r<\infty} r^{\frac{n}{pt'}} \varphi_1(r) \|\omega_1\|_{L_{pt}(B(x,r))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s} \le C\varphi_2(r). \tag{2.8}$$

Then the operator I_α is bounded from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$. From the definition of the norm of generalized weighted Morrey space we write

$$\|I_\alpha f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^{\frac{n}{qt'}} \varphi_2(r) \|\omega_2\|_{L_{qt}(B(x,r))}} \|I_\alpha f\|_{L_{q,\omega_2}(B(x,r))}. \tag{2.9}$$

We estimate $\|I_\alpha f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)}$ in (1.1) by means of Theorem 2.2 and Theorem 1.6, $\nu_2(r) = \frac{1}{\varphi_2(r)}$, $\nu_1(r) = \frac{1}{r^{\frac{n}{pt'}} \varphi_1(r)\|\omega_1\|_{L_{pt}}}$, $g(r) = \|f\|_{L_{p,\omega_1}(B(x,r))}$ and $w(s) = s^{-\frac{n}{qt'}-1}\|\omega_2\|^{-1}_{L_{qt}(B(x,s))}$, with inequality (2.8) and we obtain

$$\begin{aligned}
\|I_\alpha f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} &\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))}}{\varphi_2(r) r^{\frac{n}{qt'}} \|\omega_2\|_{L_{qt}(B(x,r))}} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s} \\
&= C \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(r)} \int_r^\infty s^{-\frac{n}{qt'}} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_{qt}(B(x,s))}} \frac{ds}{s} \\
&\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^{\frac{n}{pt'}} \varphi_1(r) \|\omega_1\|_{L_{pt}(B(x,r))}} \|f\|_{L_{p,\omega_1}(B(x,r))} \\
&= C \|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)}.
\end{aligned}$$

□

3 Conclusion

In this paper we prove the boundedness of Riesz potential I_α from the generalized weigthed Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the generalized weigthed Morrey spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$, where ω weights belong to the Fefferman-Pong class.

References

[1] Adams, D. R. (1975). A note on Riesz potentials.

[2] Aykol, C., Hasanov, J. J., & Safarov, Z. V. (2023). Two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces. Mat. Vesnik, 75(1), 37-49.

[3] Aykol, C., & Geleri, F., & Hasanov, J.J., & Safarov, Z.V. (2024). A new characterization of two-weighted inequalities for Riesz potential and its commutators in global generalized weighted Morrey spaces, submitted.

[4] Coifman, R.R., & Fefferman, C. (1974). Weighted norm inequalities for maximal functions and singular integrals, Studia Math., 51, 241-250.

[5] Guliyev, V.S. (2009) Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl., Art. ID 503948.

[6] Guliyev, V.S. (2012) Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J. 3 (3), 33-61.

[7] Guliyev, V.S. (2013) Generalized local Morrey spaces and fractional integral operators with rough kernel, J. Math. Sci. (N. Y.) 193 (2), 211-227.

[8] Morrey, C.B. (1938) On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43, 126 166.

[9] Muckenhoupt, B. & Wheeden, R. L. (1974). Weighted norm inequalities for fractional integrals, Trans. Amer. Math. Soc., 192, 261-274.

[10] Perez, C. (1994) Two weighted inequalities for potential and fractional type maximal operators, Indiana University Mathematics Journal 43 (2), 663-683.

12. HEALTH TESTS FOR CRYPTOGRAPHIC RANDOM NUMBER GENERATORS

Melis Aslan **Ali Doğanaksoy** **Zülfükar Saygı**
Fatih Sulak

Abstract

Random numbers have essential roles in cryptographic applications, they are used as keys, passwords, salts, nonce or system parameters in cryptographic algorithms and protocols. In general, these sequences are secret parts of the algorithmic process of cryptographic algorithms and protocols. The security of the cryptographic systems relies on the confidential components, as the algorithms are openly accessible. For cryptographic applications, random numbers are produced using random number generators (RNGs). There are some standards and guidelines for randomness, since designing and also validating RNGs are difficult. A RNG may be affected by outside conditions such as temperature, humidity etc. Health tests are defined to detect unexpected changes in the working process of a RNG. In this study, existing health test suites are examined and a health test suite for cryptographic RNGs is introduced.

Keywords. Randomness, Random Number, Random Number Generators, Health Tests, Entropy, Entropy Estimation, Statistical Tests, Cryptography

ORCID: 0000-0002-3964-6875
e-mail: melisa@metu.edu.tr

ORCID: 0000-0002-3055-9863
e-mail: aldoks@metu.edu.tr

ORCID: 0000-0002-7575-3272
e-mail: zsaygi@etu.edu.tr

ORCID: 0000-0002-5220-3630
e-mail: fatih.sulak@atilim.edu.tr

1 Introduction

In cryptography, random numbers have essential roles, they are used as key, password, salt nonce or system parameters in cryptographic algorithms and protocols. In general, these sequences are secret parts of the algorithmic process of cryptographic algorithms and protocols. The security of the system relies on the confidential components, as the algorithms are openly accessible.

In cryptographic contexts, random numbers are produced by random number generators (RNGs), which are categorized into two types: true random number generators (TRNGs) and pseudo-random number generators (PRNGs). TRNGs produce sequences by capturing physical phenomena that create entropy, such as radioactive decay, atmospheric noise, or electron movement. A TRNG is mainly composed of two components: a noise source or entropy source and cryptographic post processing components. Noise source extract randomness from a physical phenomena and cryptographic post processing components are designed as deterministic mathematical algorithms to increase statistical quality or randomness of the collected data in the noise soruce. On the other hand, PRNGs are deterministic algorithms and they extend seeds to long random-looking sequences.

Randomness of a sequence can be defined by three features; uniformity, independence and unpredictability.

Definition 1.1. Let S be a sequence of length n, consisting of elements from the finite set $\mathbb{A} = a_1, a_2, \cdots, a_m$. S is considered a **random sequence** if the following criteria are met:

- Every element of $\mathbb{A}$ appears in S with a probability of $\frac{1}{m}$.
- Each element of $\mathbb{A}$ is distributed in S uniformly.
- Each element of $\mathbb{A}$ is distributed in S independently.

If the outputs of RNGs do not satisfy these features, that can cause vulnerabilities. To detect such kind of defaults randomness tests are introduced.

For evaluation of outputs of TRNGs and PRNGs for cryptographic purposes statistical randomness tests are employed. In the literature, numerous randomness tests and test suites are used to evaluate the outputs of RNGs, with the most widely used being TESTU01[1], DIEHARD [2], DIEHARDER [3], and NIST SP 800-22A[4]. Moreover, for evaluation of the the other components of RNGs mainly physical components and entropy sources entropy estimation tests and health tests are employed, most widely suites constructed for this purpose are NIST SP 800-90B[5], FIPS PUB 140-2 [6] and AIS-20/AIS-31[8].

External factors such as temperature, electromagnetic interference, humidity, or mechanical vibrations can impact the functioning of an RNG. Health tests are designed to detect corruption and error in the working mechanism of the entropy source and give warnings about disruptions in the process, simultaneously. For this reason, tests should be designed as algorithms that provide fast results and have low time complexity. NIST SP 800-90B[5] and FIPS PUB 140-2 [6] recommend some health tests and describe testing process. Based on statistical randomness tests found in the literature, it is possible to define health tests to be used for these purposes. In this study, by using random variables *weight, run, runs of length 1* and *overlapping templates* we introduce an health test suite for cryptographic random number generators.

Organization. Section 2 provides descriptions of health tests in literature. In section 3, proposed health test suite is introduced with some mathematical backgrounds of random variables. Section 4 presents experimental results.

2 Literature

There are some standards and guidelines for randomness, they provide designing principles of a RNG, statistical randomness test, entropy estimation methods and health tests. NIST SP 800-90B[5] and FIPS PUB 140-2 [6] describe health tests for RNGs.

NIST SP 800-90B *Recommendation for the Entropy Sources Used for Random Bit Generation*[5] offers guidelines for the requirements of health tests and introduces two approved health tests: the Repetition Count test and the Adaptive Proportion test.

1. **Repetition Count Test:** This test identifies failures where the noise source produces the same output value over an extended period.
2. **Adaptive Proportion Test:**This test detects significant entropy loss that may arise due to physical or environmental changes. The Adaptive Proportion test checks if a particular sample appears excessively often by evaluating the frequency of the sample value within a sequence of noise source outputs.

FIPS PUB 140-2 *Security Requirements for Cryptographic Modules*[6] outlines statistical randomness tests, for a cryptographic module containing RNG. Consecutive $20,000$ bits of output of RNG is tested by the following tests:

1. **Monobit Test:** X is defined as the weight of the sequence of length $20,000$. If $9,725 < X < 10,275$ holds, then test is passed.
2. **The Poker Test:** The sequence of length $20,000$ is divided into 4-bit non-overlapping subsequences. The number of all possible 4-bit templates is 16. For each 4-bit template, let $f(i)$ denote the number of the i template in the sequence for $0 \leq i \leq 15$. Evaluate

$$X = \frac{16}{5000}(\sum_{i=0}^{15}[f(i)]^2) - 5000$$

   If $2.16 < X < 46.17$ holds, test is passed.
3. **The Runs Test:** A run refers to the longest sequence of consecutive, identical bits. For this tests, numbers of runs of the sequence of length $20,000$ are counted and stored according to their lengths. Test is passed, if the numbers of runs are in the required intervals:

Length of Run	Required Interval
1	2,343-2,657
2	1,135-1,365
3	542-708
4	251-373
5	111-201
6+	111-201

4. **The Long Run Test:** A run of length 26 or more is defined as long run. Test is passed, if there is no long run the sequence.

These tests are examined and to increase the number of tests of health test suite mathematical and statistical backgrounds of tests and random variables are analyzed in details.

3 Proposed Health Tests

By preserving similar evaluation frameworks with existing methods, a basic mathematical model of a health test suite is constructed with some selected random variables. Health test suite is created to ensure that the entropy of the random bit generator remains below the expected level during the operation, and to detect problems and faulty (mechanical or software) processes that may occur in the mechanism. The basic evaluation principle of the defined health test suite is to detect the subsequences that are at the extreme limits in the distribution functions of the selected random variables and to keep them under control.

Motivation: Let Ω_n denote the set of all binary sequences of length n, and X be a random variable, defined as $X : \Omega_n \to \mathbb{T}$ where $\mathbb{T}$ is a finite subset of non-negative integers. Assume that probability distribution function of X as follows.

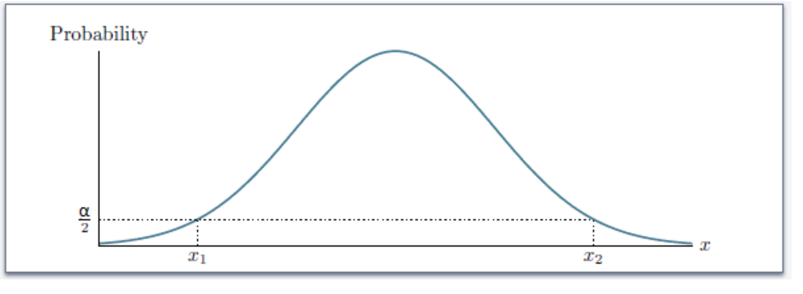


Figure 25: *Probability distribution of random variable* X

For a sequence $S \in \Omega_n$, let $X(S) = k$. For a specified significant level α, if $Pr(X = k) \in (0, \frac{\alpha}{2})$ or $Pr(X = k) \in (1 - \frac{\alpha}{2}, 1)$; that is, $k \in I = (x_1, x_2)$, the sequence S is considered at the extreme limits in the probability distribution function of the random variable X.

This test suite is designed to detect such subsequences of the output of

RNG, at the extreme limits in the probability distribution functions of the random variables *weight, run, runs of length 1* and *overlapping templates*. For each random variables, according to their probability distribution functions and significance level of the statistical test suite, valid intervals are determined.

3.1 Test Description

Some definitions and parameters are given as:
S: Binary sequence
α: Significance level (False positive probability – the likelihood that a properly functioning noise source will incorrectly fail the test for a given output.)
W: Window size (Length of each subsequence)
k: Repetition number for each test
T_i: Random variable
I_i: Valid interval for the random variable T_i

Binary sequence S is divided into k non-overlapping subsequences of length W-bit. If the length of the sequence is greater than $k.W$, remaining terms will be omitted.

This test suite is constructed by seven random variables, to preserve the significance level α for the total evaluation, totally significance levels α_i of each random variables can be evaluated by the following formula:

$$\alpha = 1 - (1 - \alpha_i)^7$$

Since each random variable is employed k-times, to test subsequences of S, for each repeated test significance levels β can be evaluated by the following formula:

$$\alpha_i = 1 - (1 - \beta)^k$$

β determines the valid intervals I_i s for each test. According to probability distribution functions of random variables, for each random variable valid intervals are determined. (This part contains some detailed calculation processes.)

The sequence S is deemed to pass test i, if for each subsequence of S, T_i is in the valid interval I_i. Algorithm of the test suite is as follows:

Algorithm 1 *Health Test Suite*

INPUT: A binary sequence S

1: Divide S into subsequences s_i s
2: **for** $i = 1$ to 7 **do**
3:     **for** $j = 1$ to k **do**
4:         **if** $T_i(s_j) \notin I_i$, "test i failed" **then**
5:         **end if**
6:     **end for**
7: **end for**
8: "pass"

3.2 Random Variables

In this section, definitions, probability distribution functions and some useful recursions of selected random variables used in the health test suite are given. For definitions and mathematical details of random variables [7] is used, more details of the random variables can be seen in [7].
Let Ω_n be the set of all binary sequences of length n. For a binary sequence $S = (s_1, s_2, \cdots, s_n)$ of length n, where $s_i \in \{0, 1\}$ for each i, a random variable is defined as $T : \Omega_n \to \mathbb{T}$ where $\mathbb{T}$ is a finite subset of non-negative integers.

3.2.1 Weight

Weight is defined as the number of 1's in the given sequence S:

$$T(S) = \sum_{i=1}^{n} s_i$$

For example; weight of the sequence $S = (0111001010011010)$ is 9.
Probability Distribution Function:

$$F_n(k) = 2^{-n} \sum_{i=0}^{k} \binom{n}{i}$$

Useful Recursions: Initial values:$F_1(0) = 1, F_1(1) = \frac{1}{2}$
For $n \geq 2$ and $k = 0$

$$F_n(0) = \frac{1}{2} F_{n-1}(0)$$

For $n \geq 2$ and $k \geq 1$

$$F_n(k) = \frac{1}{2}[F_{n-1}(k) + F_{n-1}(k-1)]$$

Whenever $k \geq n$, $F_n(k) = 1$.

3.2.2 Number of Total Runs

A run is defined as consecutive identical bits of the sequence. The number of total runs counts the runs of the given sequence S:

$$T(S) = \text{Number of total runs of } S$$

For example; the sequence $S = (0111001010011010)$ has 11 runs: 0, 111, 00, 1, 0, 1, 00, 11, 0, 1 and 0.
Probability Distribution Function:

$$F_n(k) = 2^{-n+1} \sum_{i=0}^{k} \binom{n-1}{i-1}$$

Useful Recursions: Initial values:$F_1(0) = 0, F_1(1) = 1$
For $n \geq 2$ and $k = 1$

$$F_n(0) = \frac{1}{2} F_{n-1}(0)$$

For $n \geq 2$ and $k \geq 2$

$$F_n(k) = \frac{1}{2}[F_{n-1}(k) + F_{n-1}(k-1)]$$

Whenever $k \geq n$, $F_n(k) = 1$.

3.2.3 Number of Runs of Length-1

Number of runs of length-1 counts the number of length of 1 runs of the given sequence.

$$T(S) = \text{Number of runs of Length-1} S$$

For example; the sequence $S = (0111001010011010)$ has 11 runs: 0, 111, 00, 1, 0, 1, 00, 11, 0, 1 and 0 so the number of runs of length-1 is 7.
Probability Distribution Function: Let $C_1(n, k)$ be the number of sequences of length n, containing k runs of length-1.

$$F_n(k) = 2^{-n+1} \big(\sum_{i=1}^{n} C_1(n-i, k) - C_1(n-1, k) + C_1(n-1, k-1) \big)$$

Useful Recursions:

$$C_1(n,k) = 2C_1(n-1,k) - C_1(n-1,k) + C_1(n-2,k) +$$
$$C_1(n-1,k-1) - C_1(n-2,k-1)$$

3.2.4 Overlapping Templates

This random variable counts the frequency of a predefined template of length l in the overlapping l-bit divisions of the given sequence.

$$T(S) = \text{Frequency of a predefined template in } S$$

For example; in the sequence $S = (0111001010011010)$ the template 11 of length-2 is seen 3 times.
Probability Distribution Functions: For this test suite, templates of length 4 are utilized, and their probability distribution functions are defined based on the overlapping structure of the templates as follows:
Let $T(n,k)$ be the number of sequences of length n with k overlapping substrings of length 4.

0-overlapping templates: 0001, 0011, 0111, 1000, 1100, 1110.

$$T(n,0) = 2T(n-1,0) - T(n-4,0)$$

$$T(n,k) = \begin{cases} 0 & \text{if } n < 4k \\ 1 & \text{if } n = 4k \\ 2T(n-1,k) - T(n-4,k) + T(n-4,k-1) & \text{if } n > 4k \end{cases}$$

1-overlapping templates: 0010, 0100, 1011, 1101, 0110, 1100.

$$T(n,0) = 2T(n-1,0) - T(n-3,0) + T(n-4,0)$$

$$T(n,k) = \begin{cases} 0 & \text{if } n < 3k+1 \\ 1 & \text{if } n = 3k+1 \\ 2T(n-1,k) - T(n-3,k) + & \\ T(n-4,k) + T(n-3,k-1) - & \\ T(n-4,k-1) & \text{if } n > 3k+1 \end{cases}$$

2-overlapping templates: 0101, 1010.

$$T(n,0) = 2T(n-1,0) - T(n-2,0) + 2T(n-3,0) - T(n-4,0)$$

$$T(n,k) = \begin{cases} 0 & \text{if } n < 2k+2 \\ 1 & \text{if } n = 2k+2 \\ 2T(n-1,k) - T(n-2,k) + & \\ 2T(n-3,k) - T(n-4,k) + & \\ T(n-2,k-1) - 2T(n-3,k-1) + & \\ T(n-4,k-1) & \text{if } n > 2k+2 \end{cases}$$

3-overlapping templates: : 0000, 1111.

$$T(n,0) = T(n-1,0) + T(n-2,0) + T(n-3,0) + T(n-4,0)$$

$$T(n,k) = \begin{cases} 0 & \text{if } n < k+3 \\ 1 & \text{if } n = k+3 \\ T(n-1,k) + T(n-2,k) + & \\ T(n-3,k) + T(n-4,k) + & \\ T(n-1,k-1) - T(n-2,k-1) - & \\ T(n-3,k-1) - T(n-3,k-1) & \text{if } n > k+3 \end{cases}$$

Probabilities for each i-overlapping template can be evaluated by

$$F_n(k) = \frac{T(n,k)}{2^n}$$

To construct overlapping template tests, from each i-overlapping template set a representative template is chosen instead of testing all possible templates, and test is done with four selected templates.

4 Application

4.1 Experimental Parameters and Boundary Values

To describe an application of the health test suite parameters are chosen as follows.
For the window size $W = 1024$ and repetition number $k = 976$, nearly 1 million bit of the sequence can be tested. If siginifance level of the test suite is determined as $\alpha = 2^{-33}$, for each random variable critical values are evaluated as:

$$\alpha = 1 - (1 - \alpha_i)^7$$

Then for each i, $\alpha_i = 2^{-35}$. Since each random variable is employed 976-times, for each repeated test significance levels β can be evaluated by the following formula:

$$\alpha_i = 1 - (1 - \beta)^{976}$$

Then $\beta = 2^{-45}$. By using probability distribution functions or recursions of them for each random variable, valid intervals are chosen as follows.

Random Variable	Left Boundary Value	Right Boundary Value	Significance Level ($\approx$)
Weight	445	602	2^{-45}
Run	451	606	2^{-45}
Runs of Length 1	175	353	2^{-45}
1-Overlapping Template	0	109	2^{-45}
2-Overlapping Template	0	121	2^{-45}
3-Overlapping Template	0	133	2^{-45}
4-Overlapping Template	0	180	2^{-45}

Table 8: Health Tests Boundary Values for 1 million bit

4.2 Experimental Results

The following datasets were simulated for the experiments:

1. **Uniform distribution without any known bias. AES-128** The sequences are generated by the block cipher Advanced Encryption Standard (AES) [9] using the Cipher Block Chaining (CBC) mode. This dataset contains 200 sequences of length 1 000 000. In these sequences, all outputs are assumed to have an equal probability of occurring, and are independent. Hence, they are assumed to be random.

2. **Uniform distribution without any known bias. QUANTIS** IDQ-QUANTIS [10] is a true random number generator, generation mechanism of IDQ-QUANTIS is based on quantum physics. To generate sequences, there is no need for seed or input sequence. By using IDQ-QUANTIS 200 binary sequences of length 1 000 000 are generated. The device is validated for the highest standards of entropy and randomness testing, and its outputs are presumed to be random.

3. **Biased binary distribution with Pr(0)=0.7 and Pr(1)=0.3.** The dataset follows a biased binary distribution, where the probability of observing a 0 is 0.7, and the probability of observing a 1 is 0.3. To generate this dataset, 200 sequences of length 1 000 000 were generated. This data is generated using the random number generator Mersenne Twister (MT19937) in `C++`.

4. **Biased binary t-bit Duplication** t-bit duplication is defined as copying consecutive t-bit non-overlapping blocks of the sequence to the end of each block. This method doubles the length of sequence. Sequences of length 500 000 are produced by CBC-mode of AES-128. Initial sequences are transformed with 128-bit, 256-bit and 100-bit duplication. Three datasets are generated, each contains 200 binary sequences of length 1 000 000.

		Weight	Run	Run1	Temp0	Temp1	Temp2	Temp3
AES CBC	**pass**	198	193	200	200	200	200	200
	fail	2	7	0	0	0	0	0
QUANTIS	**pass**	199	188	199	200	200	200	200
	fail	1	12	1	0	0	0	0
Biased(pr(0)=0.7)	**pass**	0	0	0	200	200	200	0
	fail	200	200	200	0	0	0	200
128-dup	**pass**	49	11	126	200	200	200	200
	fail	151	189	74	0	0	0	0
256-dup	**pass**	47	10	125	200	200	200	200
	fail	153	190	75	0	0	0	0
100-dup	**pass**	67	17	145	200	200	200	200
	fail	133	183	55	0	0	0	0

Table 9: Health tests results for simulated data sets

Health tests are employed for each dataset and numbers of passes and fails of each individual test are given in Table 2 4.2. Experimental results show that nearly all sequences of uniformly distributed datasets generated by AES-128 and IDQ-QUANTIS pass health tests as expected. The sequences of dataset follows a biased binary distribution, with the probability of observing a 0 is 0.7, and the probability of observing a 1 is 0.3, mostly failed from *weight, run* and *runs of length-1* tests. *Weight, run* and *runs of length-1* tests detect the sequences of the dataset generated by t-bit duplication. When evaluation powers of tests are compared, *template tests* can be seen weaker to detect biased sequences than *Weight, run* and *runs of length-1* tests.

5 Conclusion

Secret keys, passwords, salt, nonce are important parts of cryptographic protocols, since the security of cryptographic protocols depends on their unpredictability and randomness. That underlines the importance of generation of random number sequences. In this study, health tests for are examined, these are used to detect corruption and error in the working mechanism of the entropy source of a RNG and give warnings about disruptions in the process. Existing methods are revisited and a basic model of health test suite is introduced with random variables *weight, run, runs of length 1* and *overlapping templates*.

References

[1] L'ecuyer, P., & Simard, R. (2007). TestU01: AC library for empirical testing of random number generators. ACM Transactions on Mathematical Software (TOMS), 33(4), 1-40.

[2] Marsaglia, G. (1996). The marsaglia random number cdrom including the diehard battery of tests of randomness.

[3] Brown, R. G. (2013). Dieharder: A random number test suite.

[4] Rukhin, A., & Soto, J., & Nechvatal, J., & Smid, M., & Barker, E., & Stefan, M.L. (2001) Leigh,M. Vangel, D. Banks, A. Heckert, J. Dray, and S. Vo, A statistical test suite for random and pseudo random number generators for cryptographic applications. Technical Report.

[5] Turan, M. S., & Barker, E., & Kelsey, J., & McKay, K. A., & Baish, M. L., & Boyle, M. (2018). Recommendation for the entropy sources used for random bit generation. NIST Special Publication, 800(90B), 102.

[6] Brown, K. H. (1994). Security requirements for cryptographic modules. Fed. Inf. Process. Stand. Publ, 1-53.

[7] Koçak, O. (2016). A unified evaluation of statistical randomness tests and experimental analysis of their relations.

[8] Killmann, W., Schindler, W. (2011). A proposal for: Functionality classes for random number generators. ser. BDI, Bonn.

[9] Rijmen, V., & Daemen, J. (2001). Advanced encryption standard. Proceedings of federal information processing standards publications, national institute of standards and technology, 19, 22.

[10] https:www.idquantique.com/random-number-generation/products/quantis-random-number-generator/

13. FINITE ELEMENT ANALYSIS AND COMPARATIVE NUMERICAL EXPERIMENTS OF LERAY-α, NAVIER-STOKES-α AND NAVIER-STOKES-ω TURBULENCE MODELS

Gülnur Haçat Yılmazoğlu Aytekin Bayram Çıbık

Abstract

In this paper, we consider the semi-discrete forms of the Navier-Stokes equations regularized by the Leray-α, NS-α and NS-ω turbulence models, which play an important role in the field of fluid mechanics. The complexity and applicability of these models as well as their mathematical foundations are investigated. In particular, the validity and accuracy of the models are studied in detail by means of extensive mathematical analysis. These analyses provided an in-depth understanding of the basic principles of each model and provided important insights into whether the models accurately represent flow phenomena. Furthermore, in order to verify the practical applicability of the theoretical results obtained, numerical experiments were carried out by transferring the algorithms to a computerized environment. These numerical experiments are designed to evaluate the performance of the models in different flow structures. Thus, insights were obtained on how effective the models are in solving real-world flow problems and under which conditions they perform better. The results of this study will be useful for engineers, applied mathematicians and software vendors in developing better algorithms for numerical simulations of turbulent flows. In addition, the results will contribute to societal welfare in scientific and industrial applications, energy efficiency, polymeric material processing and biomedical device design.

Keywords. Navier Stokes equations, Turbulence models, Finite element method

ORCID: 0000-0001-7343-8466
e-mail: gulnur.yilmazoglu@yalova.edu.tr

ORCID: 0000-0003-3571-4137
e-mail: abayram@gazi.edu.tr

1 Introduction

Fluid flow problems involve complex and nonlinear partial differential equations that rarely have analytical solutions, except for simple geometries. Therefore, numerical models are crucial for approximating solutions and analyzing fluid behavior in various scenarios.

The finite element method (FEM) breaks fluid problems into smaller, manageable parts, representing each with simple mathematical expressions and then combining these parts to model the overall fluid behavior. With this approach, FEM can solve discretization problems in complex geometries, model different physical processes and produce precise results [1, 2]. Therefore, FEM was chosen as the method.

Understanding incompressible and viscous fluid flows is essential in various fields. In engineering, it enhances aerodynamic and hydrodynamic performance. In energy production, it improves fluid efficiency. In climate science, it supports research on atmospheric and ocean circulation, crucial for climate change studies, water management, disaster prediction, and biotechnology. The Navier-Stokes equations (NSE), which describe the physics of many scientifically and engineering important phenomena such as weather forecasting [3], flow in canals and pipes [4], blood flow [5], and pollution analysis [6], and express incompressible, viscous flows, are defined as follows:

$$\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= f \quad \text{in} Q, \\ \nabla\cdot y &= 0 \quad \text{in} Q, \\ y &= 0 \quad \text{on}(0,T)\times\partial\Omega, \\ y(0,x) &= y_0 \quad \text{in}\Omega, \end{aligned} \tag{1.1}$$

Here Ω is a bounded and regular flow region defined in $\mathbb{R}^d (d = 2, 3)$ and

$Q = [0, T] \times \Omega$ for $T > 0$. NSD deals with fluid dynamics in detail by calculating the effects of fluid velocity (y), pressure (p), force (f) and viscosity (ν). Relative to viscosity, the Reynolds number (Re) provides insight into the behavior of the flow. At high Re numbers ($Re > 4000$) the convective term ($(y \cdot \nabla)y$) becomes dominant, leading to turbulent flow.

Turbulent flows, characterized by irregular motions and a wide range of flow scales, pose a challenge for numerical calculations of the NSE. Small scales are crucial for accurately capturing turbulent behavior. Therefore, a turbulence model is necessary to account for these effects in numerical simulations. Different approaches have been proposed for the simulation of turbulent flows, such as Direct Numerical Simulation (DNS), Reynolds Averaged Navier-Stokes (RANS), Large Eddy Simulation (LES), Leray-α, Navier-Stokes-α (NS-α) and Navier-Stokes-ω (NS-ω) models, in an effort to alter the NSE to more closely resemble the averages of flow structures rather than the real flow.

The large number of turbulence models used for numerical solutions of NSE makes the choice of simulation model difficult. For the solution of the turbulence model to be appropriate, the model should share the same physical properties as the NSE. For this purpose, Leray-α [7, 8, 9, 10], NS-α [11, 12, 13, 14] and NS-ω [15, 16] models are developed as smoothed forms of NSE.

In section 3, the mathematical and physical validity of the NSE organized by the Leray-α turbulence model is investigated. It is shown that the stability and convergence are verified by mathematical analysis as a result of temporal and spatial discretization of the obtained solutions. In section 4, numerical analysis of the NSE organized with NS-α and NS-ω turbulence models is presented. The stability and convergence of the method are analyzed. In section 5, several numerical examples are carried out to support the theoretical results obtained and to demonstrate the efficiency and accuracy of the models. Finally, section 6 presents the conclusions of the study and discussions on possible future research directions.

2 Mathematical Foundations

Definition 2.1. Lebesgue Spaces. Lebesgue spaces are the class of all measurable functions whose p-th powers are integrable and are defined as follows:

$$L^p(\Omega) = \Bigg\{ f : f, \text{is a measurable function and for } \forall\, 1 \leq p < \infty,$$

$$\int_\Omega |f(t)|^p dx < \infty \Big\}.$$

Lebesgue spaces, denoted as L^p spaces, are crucial in mathematical analysis because they generalize the concept of integrability, allowing for a broader class of functions to be studied. Unlike classical spaces that only handle continuous or differentiable functions, Lebesgue spaces encompass measurable functions, even those with discontinuities or singularities, provided they meet certain integrability conditions. This makes them particularly useful for working with functions that may not behave well everywhere but are still integrable in a more generalized sense. By focusing on the p-th powers of these functions, Lebesgue spaces offer a flexible framework for capturing different levels of smoothness or decay, making them suitable for applications in various fields, such as physics and partial differential equations. Additionally, these spaces are fundamental for studying the convergence of sequences of functions, which is key in many areas of analysis where classical pointwise convergence may fail. Overall, the definition of Lebesgue spaces is essential for extending the idea of integration and providing the tools necessary for working with more complex functions in modern analysis.

Remark 2.2. The norm $L^p(\Omega)$ is defined as follows:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

Since this space is a Hilbert space, an important special case gives $L^2(\Omega)$ for $p = 2$. The inner product and norm of $L^2(\Omega)$ are as follows:

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \qquad \|f\|_{L^2(\Omega)} = (f, f)^{1/2}_{L^2(\Omega)}.$$

The space of functions bounded for nearly all $x \in \Omega$ is denoted as $L^\infty(\Omega)$:

$$L^\infty(\Omega) = \Big\{ f : \text{for almost all } x \in \Omega \text{ is } |f(x)| < \infty \Big\}.$$

Definition 2.3. Sobolev Space $W^{k,p}(\Omega)$. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of all integrable functions $f : \Omega \to \mathbb{R}^d$ so that $|\alpha| \leq k$ for each multiple index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, where $D^\alpha f$ defines the weak derivative of order $|\alpha|$ and belongs to $L^p(\Omega)$:

$$W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for } |\alpha| \leq k \}.$$

This means that not only must f itself be integrable, but all its weak derivatives up to the specified order k should also satisfy the integrability condition, ensuring that they are p-integrable over the domain Ω.

Sobolev spaces are essential in mathematics and engineering for several important reasons. They allow for the weak differentiation of functions when classical derivatives are undefined or do not exist, providing the means to differentiate over a broader class of functions. Sobolev spaces combine specific regularity and integrability requirements, helping to determine how well a function behaves with respect to a given p-norm. They play a fundamental role in analyzing solutions to partial differential equations, allowing for the examination of properties such as existence, uniqueness, and continuity of solutions. Additionally, Sobolev spaces are crucial in functional analysis, particularly in analyzing operators and boundary value problems. They are also used in optimization problems and control theory applications, aiding in the understanding of regularity properties of controlled systems.

Remark 2.4. For $k = 0$, $L^p(\Omega) = W^{0,p}(\Omega)$. A norm in Sobolev space is defined as follows:

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} p \in [1,\infty] \text{ ise }, \left(\sum\limits_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \\ p = \infty \text{ ise }, \sum\limits_{|\alpha|\leq k} \operatorname*{ess\,sup}_{x\in\Omega} |D^\alpha f|. \end{cases}$$

First-order Sobolev spaces, which are important for the study of NSE:

$$W^{1,p}(\Omega) = \left\{ f : \int_\Omega (|f(x)|^p + |\nabla f(x)|^p) dx < \infty \right\},\ p \in [1,\infty),$$

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|f(x)|^p + |\nabla f(x)|^p) dx \right)^{1/p},\ p \in [1,\infty).$$

The Sobolev space used in this study is the following closed subspace of $H^1(\Omega)$:

$$H_0^1(\Omega) := \{ v \in H^1(\Omega) : \partial\Omega \text{ üzerinde } v = 0 \}.$$

The H^{-1} dual space of $H_0^1(\Omega)$ has the following norm:

$$\|f\|_{-1} = \sup_{v\in H_0^1(\Omega)} \frac{(f,v)}{\|\nabla v\|}.$$

Definition 2.5. For any (scalar or vector-valued) function $v(x,t)$ defined on $\Omega \times (0,T]$ for a given finite finite time $T > 0$, the following norms are used:

$$\|v\|_{\infty,k} := \operatorname*{ess\,sup}_{0\leq t\leq T} \|v(\cdot,t)\|_k\ , \qquad \|v\|_{p,k} = \left(\int_0^T \|v(\cdot,t)\|_p^k dx \right)^{1/p}.$$

Discrete norms are defined with the following notations:

$$|||v|||_{\infty,k} := \operatorname*{ess\,sup}_{0\leq n\leq N} \|v^n\|_k \ , \qquad |||v|||_{p,k} = \left(\Delta t \sum_{n=0}^{N} \|v^n\|_p^k dx\right)^{1/p} .$$

Here Δt is the time step such that $t^n = n\Delta t(n = 0, 1, ..., N)$ and $v(t^n) = v^n$.

In the finite element approximation of the NSE given in (1.1), the velocity $\mathbf{Y} = \left(H_0^1(\Omega)\right)^2$ and pressure continuous spaces $M = L_0^2(\Omega)$ are chosen. The set of divergence-free functions in $\mathbf{Y}$ is defined as follows:

$$\mathbf{V} := \{(\nabla \cdot v, q) = 0 \text{ for } v \in \mathbf{Y} : \ \forall q \in M\}.$$

Multiplying (1.1) by the test functions $\forall(v, q) \in (\mathbf{Y}, M)$ and integrating over the Ω region and applying Green's theorem yields the variational formulation as follows:

$$\begin{aligned} (y_t, v) + \nu\left(\nabla y, \nabla v\right) + ((y\cdot\nabla)y, v) - (p, \nabla\cdot v) &= (f, v)\,, \\ (\nabla\cdot y, q) &= 0. \end{aligned} \tag{2.6}$$

Here $y : (0, T] \to \mathbf{Y}$ and $p : (0, T] \to M$. The convective term in (2.6) for $\forall y, v, w \in \mathbf{Y}$ is defined in inverse-symmetric trilinear form as follows [17, 18]:

$$b(y, v, w) = \frac{1}{2}\left(((y\cdot\nabla)v, w) - ((y\cdot\nabla)w, v)\right), \tag{2.7}$$

$$b(y, v, w) = ((\nabla v)^T w, y). \tag{2.8}$$

From the given definitions it is clear that $b(y, v, v) = 0$ and $b(y, v, w) = -b(y, w, v)$. Error analysis will require the following upper bounds for the inverse-symmetric trilinear form (2.7).

Lemma 2.9. [19] For $y, v, w \in \mathbf{Y}$, the inverse-symmetric trilinear form $b(\cdot, \cdot, \cdot)$ satisfies the following upper bound:

$$|b\left(y, v, w\right)| \leq C\|\nabla y\|\|\nabla v\|\|\nabla w\|, \tag{2.10}$$

Here C is a constant that depends only on Ω. This inequality indicates that the magnitude of the trilinear form can be controlled by the norms of the involved functions, ensuring boundedness based on the properties of the space. Also, for $v, \nabla v \in L^\infty(\Omega)$ we have

$$|b\left(y, v, w\right)| \ \leq \ \frac{1}{2}(\|y\|\|\nabla v\|_\infty\|w\| + \|y\|\|v\|_\infty\|\|\nabla w\|). \tag{2.11}$$

For the finite element discretization, a family of triangulations, denoted as $\zeta_h = \{K_j\}_{j=1}^{M}$, is employed. Suppose the computational domain Ω is partitioned into ζ_h, where each element in the mesh forms a quasi-uniform simplex configuration. Specifically, each triangle $K_j \in \zeta_h$ is associated with a discretization parameter, h, which is defined as the maximum diameter h_K across all the triangles in the mesh:

$$h = \max_{K_j \in \zeta_h} h_K$$

where h_K represents the diameter of the specific element K_j.

The neighborhood of a node, denoted as $\varsigma_h(x) \in \Omega$ for each point x within the mesh ζ_h, comprises all the cells K_j that share the node $x \in \partial K_j$. That is, the neighborhood of a node is made up of all elements adjacent to the node x.

Complying finite element subspaces $\mathbf{Y}^h \subset \mathbf{Y}$ and $M^h \subset M$ are selected to approximate the velocity and pressure variables, respectively, while ensuring that the discrete inf-sup condition is satisfied:

$$\inf_{q^h \in M^h} \sup_{v^h \in \mathbf{Y}^h} \frac{(\nabla \cdot v^h, q^h)}{\|\nabla v^h\| \|q^h\|} \geq \beta > 0, \tag{2.12}$$

where β is independent of the mesh size h. This condition is critical for the stability of the finite element approximation, particularly when solving problems involving fluid dynamics or incompressible flow.

It is well established that the Taylor-Hood and mini element pairs satisfy the condition referenced in equation (2.12), as discussed in sources such as [9, 20]. These pairs are highly regarded for their ability to fulfill the necessary conditions in various finite element formulations. By utilizing piecewise polynomial functions of degree k for velocity and $k-1$ for pressure, one can achieve the desired convergence properties. Consequently, the spaces $(\mathbf{Y}^h, M^h)$, which represent the finite-dimensional subspaces for velocity and pressure respectively, possess widely accepted approximation capabilities, as outlined by the standard finite element theory:

$$\inf_{v^h \in \mathbf{Y}^h} \left(\|(y - v^h)\| + h \|\nabla (y - v^h)\| \right) \leq C h^{k+1} \|y\|_{k+1} \quad y \in H^{k+1}(\Omega), \tag{2.13}$$

$$\inf_{q^h \in M^h} \|p - q^h\| \leq C h^k \|p\|_k \quad p \in H^k(\Omega), \tag{2.14}$$

for $(v^h, q^h) \in (\mathbf{Y}^h, M^h)$. The discretely divergence free subspace of $\mathbf{Y}^h$ is defined by

$$\mathbf{V}^h = \{v^h \in \mathbf{Y}^h : (\nabla \cdot v^h, q^h) = 0, \forall q^h \in M^h\}. \tag{2.15}$$

Under the inf-sup condition (2.12), it is known that the weak formulations of NSE in $\mathbf{Y}^h$ and in $(\mathbf{V}^h)$ are equivalent.

The Discrete Gronwall's Inequality is a fundamental tool in the analysis of numerical methods for differential equations, particularly in stability and convergence proofs. The Discrete Gronwall's Inequality is essential for proving the reliability, stability, and accuracy of numerical methods, ensuring that the solutions to discrete problems behave similarly to the solutions of the continuous equations they approximate.

Lemma 2.16. **Discrete Gronwall's Inequality** Let k, B, and the sequences a_n, b_n, c_n and d_n be non-negative numbers for integers $n \geq 1$. Accordingly, assume that the inequality

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \leq B + k\sum_{n=1}^{N+1} c_n + k\sum_{n=1}^{N} d_n a_n \qquad N \geq 0,$$

is satisfied. If $kd_n < 1$ for $\forall\, n = 1, ..., N+1$, then

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \leq \left(B + k\sum_{n=1}^{N+1} c_n\right) exp\left(k\sum_{n=1}^{N} d_n\right) \qquad N \geq 0,$$

is satisfied [21].

Throughout the analysis, the vector identities listed below will be used. Assume that a, b, and c are elements belonging to a set X, where $a(x), b(x)$, and $c(x)$ are vector fields in $\mathbb{R}^3$ for every point x in the domain Ω^d. Under these conditions, the subsequent vector identities hold:

$$\begin{aligned} (\nabla \times a) \times b &= (b \cdot \nabla)a - \nabla(a \cdot b) + (\nabla b)^T a, \\ ((\nabla \times a) \times b, c) &= ((b \cdot \nabla)a, c) - ((c \cdot \nabla)a, b). \end{aligned} \tag{2.17}$$

Lemma 2.18. For $a, b, c \in X$, or $L_\infty(\Omega)$ and $\nabla \times a \in L_\infty(\Omega)$ when indicated, the trilinear term $((\nabla \times a) \times b, c)$ satisfies

$$\begin{aligned} |((\nabla \times a) \times b, c)| &\leq \|\nabla \times a\| \|b\|_\infty \|c\|, \\ |((\nabla \times a) \times b, c)| &\leq \|\nabla \times a\|_\infty \|b\| \|c\|, \\ |((\nabla \times a) \times b, c)| &\leq C\|\nabla \times a\| \|\nabla b\| \|\nabla c\|, \\ |((\nabla \times a) \times b, c)| &\leq C\|b\|^{1/2} \|\nabla b\|^{1/2} \|\nabla \times a\| \|\nabla c\|, \\ |((\nabla \times a) \times b, c)| &\leq C\|c\|^{1/2} \|\nabla c\|^{1/2} \|\nabla a\| \|\nabla b\|, \end{aligned} \tag{2.19}$$

Proof. The proof can be found in e.g. [16]. □

3 NSE Regulated with Leray-α Model

The idea of the Leray-α model is derived from Leray's pioneering work [22]. In [22], the existence of a variational solution of the nonlinear term of the NSE was proved by replacing the convection field by a regular velocity field, i.e. by using $(\overline{y}\cdot\nabla)y$ instead of $(y\cdot\nabla)y$ in (1.1). For a $\theta \in L^2(\Omega)$, $-\alpha^2\Delta\overline{\theta}+\overline{\theta} = \theta$ for a $\theta \in L^2(\Omega)$ with $\alpha > 0$, $\overline{\theta}$ is called a α radius filter for θ. The Leray-α model to be analyzed depending on the filter is as follows:

$$\begin{aligned} y_t - \nu\Delta y + (\overline{y}\cdot\nabla)y + \nabla p &= f, \\ \nabla\cdot y &= 0. \end{aligned} \tag{3.1}$$

The basic logic of this model is to obtain a smoothed system by writing the NSE for y^α for a given value of α. Fix $\alpha > 0$ and set $\overline{y}^\alpha = (I-\alpha^2\Delta)^{-1}y^\alpha$ to obtain the

$$\begin{aligned} y_t - \nu\Delta y + (\overline{y}\cdot\nabla)y + \nabla p = f \qquad & (t,x) \in \Omega\times(0,T], \\ \overline{y} - \alpha^2\Delta\overline{y} = y \qquad & (t,x) \in \Omega\times(0,T], \end{aligned} \tag{3.2}$$

system under periodic boundary conditions [7].

Definition 3.3. Continuous Differential Filter Let $y \in L^2(\Omega)$ where y represents a function in the L^2 space over the domain Ω, and let α be a predefined filtering radius. Then, the unique solution $y \in \mathbf{Y}$ that satisfies the following equation is referred to as a continuous filter of y. This filtered version of y smooths out high-frequency variations, depending on the chosen radius α, and produces a more regular, less oscillatory version of the function.

$$\alpha^2(\nabla\overline{y},\nabla v) + (\overline{y},v) = (y,v), \qquad \forall v \in \mathbf{Y}.$$

Definition 3.4. Discrete Differential Filter Let $y \in L^2(\Omega)$, where y is a function belonging to the L^2 space over the domain Ω, and let α denote a selected filtering radius. The discrete filter of y, denoted as $\overline{y}^h \in \mathbf{Y}^h$, is the unique solution of the following equation. This discrete filter acts to smooth the function y at a resolution determined by the chosen mesh or discretization parameter h, using the filtering radius α to control the extent of the smoothing process in a finite-dimensional space.

$$\alpha^2(\nabla\overline{y}^h,\nabla v^h) + (\overline{y}^h,v^h) = (y,v^h), \qquad \forall v^h \in \mathbf{Y}^h.$$

The stability and error estimates of the discrete differential filter will be useful in the analysis of the turbulence models studied [9].

Lemma 3.5. **Stability of a Discrete Differential Filter** Let $y \in \mathbf{Y}$, then the inequalities

$$\|\overline{y}^h\| \leq \|y\| \qquad \text{ve} \qquad \|\nabla \overline{y}^h\| \leq \|\nabla y\|. \tag{3.6}$$

are obtained [9].

Lemma 3.7. **Error Estimation for Discrete Differential Filter** Let $y \in \mathbf{Y}$ with $\Delta y \in L^2(\Omega)$, then we get

$$\|y - \overline{y}^h\|^2 + \alpha^2 \|\nabla(y - \overline{y}^h)\|^2 \leq C \inf_{v^h \in \mathbf{Y}^h} \{\|y - v^h\|^2 \tag{3.8}$$

$$+\alpha^2 \|\nabla(y - v^h)\|^2\} + C\alpha^4 \|\Delta y\|^2,$$

Here C is a constant that is independent of α and h. Hence

$$\|\nabla(y - \overline{y}^h)\| \leq C\alpha^{-1} \left((\alpha + h)\|\nabla y\| + \alpha^2 \|\Delta y\|\right), \tag{3.9}$$

$$\|y - \overline{y}^h\| \leq C \left((\alpha + h)\|\nabla y\| + \alpha^2 \|\Delta y\|\right). \tag{3.10}$$

In filter-based stabilization methods, the determination of the α filter parameter varies according to the problem used. These filter parameters are used as stabilizing elements in the system and their optimal values are determined based on preliminary results obtained by convergence analysis and numerical tests. For example, in [23], the Leray-α model was applied to the NSE and the convergence analysis and numerical tests showed that the optimal value of α is $\alpha = Ch$ (C is the filter thickness constant).

3.1 Numerical Analysis

The semi-discretized finite element formulation of the Leray-α model, where time remains continuous while space is discretized, is presented as follows. This approach involves discretizing the spatial domain using finite elements while keeping the temporal aspect continuous, allowing for a detailed numerical approximation of the Leray-α model, which is often used in fluid dynamics to model regularized flow behavior at various scales.

For $y^h \in \mathbf{Y}^h$, $p^h \in M^h$ and $\forall (v^h, q^h) \in (\mathbf{Y}^h, M^h)$:

$$(y_t^h, v^h) + \nu(\nabla y^h, \nabla v^h) + b(\overline{y^h}^h, y^h, v^h) - (p^h, \nabla \cdot v^h) = (f^h, v^h), \tag{3.11}$$

$$(\nabla \cdot y^h, q^h) = 0, \tag{3.12}$$

$$(\overline{y^h}^h, v^h) + \alpha^2 (\nabla \overline{y^h}^h, \nabla v^h) = (y^h, v^h), \tag{3.13}$$

Lemma 3.14. **Stability analysis.** Let $y^h(0) \in \mathbf{Y}^h$ and $u^h \in L^2(0,T;H^{-1})^2$. The state variable then suggests stability of the equation of state by satisfying the following condition.

$$\|y^h(t)\|^2 + \nu \int_0^t \|\nabla y^h(s)\|^2 ds \leq \|y^h(0)\|^2 + \nu^{-1} \int_0^t \|u^h(s)\|_{-1}^2 ds.$$

Proof. The proof starts by choosing $v^h = y^h$ as the test function in (3.11) and $q^h = p^h$ as the test function in (3.12) and applying the Cauchy-Schwarz inequality. The duality estimate, Young's inequality, and the nonlinear term's inverse symmetry round up the proof. The proof's specifics are contained in Lemma 7.34 in [9]. □

Lemma 3.6 provides information about the discrete differential filter's stability.

The following regularity assumptions are assumed to hold for continuous solutions.

$$\begin{aligned} y \in L^\infty(0,T;H^1(\Omega)^2) \cap H^1(0,T;H^{k+1}(\Omega)^2) \cap H^3(0,T;L^2(\Omega)^2) \\ \cap H^2(0,T;H^1(\Omega)^2), \\ p \in L^2(0,T;H^{s+1}(\Omega)^2) \cap H^2(0,T;L^2(\Omega)^2), \\ f \in L^2(0,T;L^2(\Omega)^2). \end{aligned} \tag{3.15}$$

Lemma 3.16. **Error analysis.** Regarding error analysis, it is expected that assumption (3.15) is satisfied. Then, the boundary of the error $y - y^h$ is as follows [9]:

$$\|y - y^h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu \|\nabla(y - y^h\|_{L^2(0,T;L^2(\Omega))}^2 \leq E_y.$$

Here,

$$\begin{aligned} E_y = \exp\left(C\int_0^T \|\nabla y\|^4\right) \Bigg[\|y_0 - y^h(0)\|^2 \\ + \inf_{\tilde{y} \in \mathbf{Y^h}} \Bigg\{ \int_0^T \Bigg((\nu + \|y^h\|^2)\|\nabla(y - \tilde{y})\|^2 \\ + h^{-1}\nu^{-1}\|y - \tilde{y}\|^2 + \nu^{-1}\|(y - \tilde{y})_t\|^2 + \nu^{-1}\|p - p^h\|^2 \\ + \nu^{-1}\left((\alpha + h)^2\|\nabla y\|^2 + \alpha^4\|\Delta y\|^2\right)(\|y\|_\infty + \|\nabla y\|_\infty)^2 \Bigg) dt \Bigg\} \Bigg]. \end{aligned}$$

Proof. (3.11) is subtracted from (1.1) to obtain

$$(e_t, v^h) - \nu(\nabla e, \nabla v^h) + b(y, y, v^h) - b(\overline{y^h}^h, y^h, v^h) + (p - p^h, \nabla \cdot v^h) \\ = 0.$$

Here the error $e = y - y^h$ is decomposed as $e = y - y^h = y - \tilde{y} - (y^h - \tilde{y}) = \eta - \phi^h$. $\tilde{y}$ is the arbitrary interpolation of y in $\mathbf{V}^h$. If we choose the test function $v^h = \phi^h$, we get

$$\frac{1}{2}\frac{d}{dt}\|\phi_t^h\|^2 + \nu\|\nabla\phi^h\|^2 = -b(y, y, \phi^h) + b(\overline{y^h}^h, y^h, \phi^h) - (p - p^h, \nabla \cdot \phi^h) \\ + \nu(\nabla\eta, \nabla\phi^h) - (\eta_t, \phi^h). \tag{3.17}$$

Rearranging the non-linear terms in (3.17) yields

$$-b(y, y, \phi^h) + b(\overline{y^h}^h, y^h, \phi^h) \\ = -b(y - \overline{y}^h, y, \phi^h) + b(\overline{\eta}^h, y, \phi^h) + b(\overline{\phi^h}^h, y, \phi^h) - b(\overline{y^h}^h, \eta, \phi^h). \tag{3.18}$$

Through Cauchy-Schwarz inequality, Young inequality and Lemma 2.9,

$$b(y - \overline{y}^h, y, \phi^h) \le C\nu^{-1}\|y - \overline{y}^h\|^2 \left(\|\nabla y\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)}\right)^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2 \tag{3.19}$$

$$\le C\nu^{-1}\left((\alpha + h)^2\|\nabla y\|^2 + \alpha^4\|\Delta y\|^2\right)\left(\|\nabla y\|_{L^\infty(\Omega)} + \|y\|_{L^\infty(\Omega)}\right)^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2,$$

$$b(\overline{y^h}^h, \eta, \phi^h) \le C\nu^{-1}\|\overline{y^h}^h\|\|\nabla(\overline{y^h}^h)\|\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2 \\ \le C\nu^{-1}\|\nabla y^h\|^2\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2 \tag{3.20}$$

$$b(\overline{\eta}^h, y, \phi^h) \le C\nu^{-1}\|\eta\|\|\nabla\eta\|^2\|\nabla y\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2, \tag{3.21}$$

$$b(\overline{\phi^h}^h, y, \phi^h) \le C\nu^{-3}\|\phi^h\|^2\|\nabla y\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2, \tag{3.22}$$

inequalities are obtained. Using the Cauchy-Schwarz, Young and Poincare inequalities, we obtain the following result:

$$(p - p^h, \nabla \cdot \phi^h) \le C\nu^{-1}\|p - p^h\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2, \tag{3.23}$$

$$\nu(\nabla\eta, \nabla\phi^h) \le C\nu\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2 \tag{3.24}$$

$$(\eta_t, \phi^h) \le C\nu^{-1}\|\eta_t\|^2 + \frac{\nu}{14}\|\nabla\phi^h\|^2. \tag{3.25}$$

All the obtained boundaries are added to (3.17). Finally, Lemma is produced by applying the triangle inequality to the integral $[0, T]$.

□

4 NSE Regulated with NS-α and NS-ω Models

Both models belong to the family of Large Eddy Simulation (LES) models and predicts the larger scales of fluid flow on much coarse meshes successively.

The first (NS-α) model is also known as viscous Camassa-Holm (CH) equation takes the form

$$\begin{aligned} y_t - \nu\Delta y + (\nabla \times y) \times \overline{y} + \nabla P &= f \quad \text{in } Q, \\ \nabla \cdot \overline{y} &= 0 \quad \text{in } Q, \\ \overline{y} - \alpha^2 \Delta \overline{y} &= y \quad \text{in } Q. \end{aligned} \tag{4.1}$$

The second model, a complement of NS-α is called NS-ω and is obtained by averaging vorticity ω=$\nabla \times \overline{y}$ in (4.1) and given by

$$\begin{aligned} y_t - \nu\Delta y + (\nabla \times \overline{y}) \times y + \nabla P &= f \quad \text{in } Q, \\ \nabla \cdot y &= 0 \quad \text{in } Q, \\ \overline{y} - \alpha^2 \Delta \overline{y} &= y \quad \text{in } Q. \end{aligned} \tag{4.2}$$

The choice between NS-α and NS-ω models depends on application type and flow conditions, as detailed in [16]. The semi-discrete numerical schemes obtained with the studied FEM are defined as follows.

For $y^h \in \mathbf{Y}^h, P^h \in M^h$, and $(v^h, q^h) \in (\mathbf{Y}^h, M^h)$:

NS-α model:

$$(y_t, v^h) + \nu(\nabla y^h, \nabla v^h) + ((\nabla \times y^h) \times \overline{y^h}^h, v^h) - (P^h, \nabla \cdot v^h) = (u^h, v^h), \tag{4.3}$$

$$(\overline{y^h}^h, v^h) + \alpha^2(\nabla \overline{y^h}^h, \nabla v^h) = (y^h, v^h), \tag{4.4}$$

NS-ω model:

$$(y_t, v^h) + \nu(\nabla y^h, \nabla v^h) + ((\nabla \times \overline{y^h}^h) \times y^h, v^h) - (P^h, \nabla \cdot v^h) = (u^h, v^h), \tag{4.5}$$

$$(\overline{y^h}^h, v^h) + \alpha^2(\nabla \overline{y^h}^h, \nabla v^h) = (y^h, v^h). \tag{4.6}$$

4.1 Numerical Analysis

Lemma 4.7. **Stability analysis** The equations of state for both the NS-α and NS-ω models give the following inequality:

$$\|\overline{y^h(t)}^h\|^2 + \alpha^2\|\nabla \overline{y^h(t)}^h\|^2 + \nu \int_0^T \left\{ \|\nabla \overline{y^h(t)}^h\|^2 + \alpha^2 \|\Delta \overline{y^h(t)}^h\|^2 \right\} dt \leq C,$$

$$\|y^h(t)\|^2 + \nu \int_0^T \|\nabla y^h(t)\|^2 dt \leq C.$$

Here $C = C\left(y^h(0), f, \nu\right)$.

Proof. The proof starts by choosing $v^h = \overline{y^h}^h$ as the test function in (4.3). Then, the appropriate identity in (2.17) is used for the nonlinear term. Hölder and Young inequality are used in the resulting equation. For the second inequality in Lemma 4.7, it is sufficient to use the assumptions in Lemma 3.6. $\square$

Since both models give similar results, only the NS-α model is used in the error analysis.

Lemma 4.8. **Error analysis** (1.1) ve (4.1)'nin çözümleri sırasıyla y ve y^h olsun. O zaman, pozitif bir C sabiti vardır, öyle ki $y - y^h$ hatası için aşağıdaki sınır geçerlidir:

$$\|y - y^h\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla(y - y^h\|^2_{L^2(0,T;L^2(\Omega))} \leq E_y$$

where,

$$\begin{aligned}E_y &= \exp\left(C\nu^{-3}\|\nabla y\|^4\right)\Big[\|y - y^h(0)\|^2_{L^\infty(0,T;L^2(\Omega))}\\ &+ C\inf\Big\{\int_0^T\left(\nu^{-1}\|(y-\tilde{y})_t\|^2 + \nu\|\nabla(y-\tilde{y})\|^2 + \nu^{-1}\|p - P^h\|^2\right.\\ &\left.+ \nu^{-1}\|\nabla(y-\tilde{y})\|^2\|\nabla y\|^2 + \nu^{-1}\|\nabla(y-\tilde{y})\|^2\|\nabla y^h\|^2 + \nu^{-1}\alpha^4\|\nabla y\|^2_\infty|y|^2_2\right)dt\Big\}\Big].\end{aligned}$$

Proof. This inequality is obtained by subtracting (4.1) from the weak formulation of the rotation NSE. For NS-α:

$$\begin{aligned}(e_t, v_h) + \nu(\nabla e, \nabla v_h) + ((\nabla\times y)\times y, v_h) - ((\nabla\times y_h \times \overline{y_h}^h, v_h)\\ -(p - P_h, \nabla\cdot v_h) = 0,\end{aligned}$$

where $e = y - y_h$. By taking $e = y - y_h = y - \tilde{y} - (y_h - \tilde{y}) = \eta - \phi_h$ with the best approximation of y and choosing the test function $v_h = \phi_h$, the following equation is obtained:

$$\begin{aligned}(\phi_t, \phi_h) + \nu(\nabla\phi_h, \nabla\phi_h) = (\eta_t, \phi_h) + \nu(\nabla\eta, \nabla\phi_h) + ((\nabla\times y)\times y, \phi_h)\\ - ((\nabla\times y_h\times\overline{y_h}^h, \phi_h) - (p - P_h, \nabla\cdot\phi_h).\end{aligned}$$

For nonlinear terms,

$$\begin{aligned}((\nabla\times y)\times y, \phi_h) - ((\nabla\times y_u\times\overline{y_h}^h, \phi_h) &= ((\nabla\times(y - y_h)\times y, \phi_h)\\ &+ ((\nabla\times y_h\times(y - \overline{y_h}^h), \phi_h)\\ &= ((\nabla\times\eta)\times y, \phi_h)\\ &- ((\nabla\times\phi_h)\times y, \phi_h)\\ &+ ((\nabla\times y_h\times(y - \overline{y_h}^h), \phi_h).\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \nu\|\nabla\phi_h\|^2 = (\eta_t, \phi_h) + \nu(\nabla\eta, \nabla\phi_h) - (p - P_h, \nabla\cdot\phi_h) \\ + ((\nabla\times\eta)\times y, \phi_h) - ((\nabla\times\phi_h)\times y, \phi_h) \\ + ((\nabla\times y_h \times (y - \overline{y_h}^h), \phi_h).$$

The terms on the right are bounded as:

$$\begin{aligned}
(\eta_t, \phi_h) &\leq C\nu^{-1}\|\eta_t\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\
\nu(\nabla\eta, \nabla\phi_h) &\leq C\nu\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\
(p - P_h, \nabla\cdot\phi_h) &\leq C\nu^{-1}\|p - P_h\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\
((\nabla\times\eta)\times y, \phi_h) &\leq C\nu^{-1}\|\nabla\eta\|^2\|\nabla y\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\
((\nabla\times\phi_h)\times y, \phi_h) &\leq C\nu^{-3}\|\phi_h\|^2\|\nabla y\|^4 + \frac{\nu}{14}\|\nabla\phi_h\|^2
\end{aligned}$$

$$\begin{aligned}
& ((\nabla\times y_h \times (y - \overline{y_h}^h), \phi_h) \\
= \; & ((\nabla\times y_h \times (y - \overline{y}^h), \phi_h) + ((\nabla\times y_h \times (\overline{y}^h - \overline{y_h}^h), \phi_h) \\
= \; & ((\nabla\times y_h \times (y - \overline{y}^h), \phi_h) + ((\nabla\times y_h \times \overline{\eta}^h, \phi_h) \\
= \; & T_1 + T_2
\end{aligned}$$

For T_1,

$$\begin{aligned}
((\nabla\times y_h \times (y - \overline{y}^h), \phi_h) &\leq \|\nabla\times y_h\|_\infty \|y - \overline{y}^h\|\|\nabla\phi_h\| \\
&\leq C\nu^{-1}\|\nabla\times y_h\|_\infty^2 \|y - \overline{y}^h\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\
&\leq C\nu^{-1}\alpha^4\|\nabla\times y\|_\infty^2 |y|_2^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2.
\end{aligned}$$

For T_2,

$$\begin{aligned}
((\nabla\times y_h \times \overline{\eta}^h, \phi_h) &\leq C\|\nabla y_h\|\|\nabla\overline{\eta}^h\|\|\nabla\phi_h\| \\
&\leq C\nu^{-1}\|\nabla y_h\|^2\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2
\end{aligned}$$

Combining all these results, we can rewrite equation as follows:

$$\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \frac{\nu}{2}\|\nabla\phi_h\|^2 \leq C\Big(\nu^{-1}\|\eta_t\|^2 + \nu\|\nabla\eta\|^2 + \nu^{-1}\|p - P_h\|^2 \\ + \nu^{-1}\|\nabla\eta\|^2\|\nabla y\|^2 + \nu^{-3}\|\phi_h\|^2\|\nabla y\|^4 \\ + \nu^{-1}\alpha^4\|\nabla y\|_\infty^2 |y|_2^2 + \nu^{-1}\|\nabla y_h\|^2\|\nabla\eta\|^2\Big).$$

We integrate over [0, T]:

$$\|\phi_h(T)\|^2 + \nu \int_0^T \|\nabla \phi_h\|^2 dt \leq \|\phi_h(0)\|^2 + C \int_0^T \Big(\nu^{-1} \|(y-\tilde{y})_t\|^2 + \nu \|\nabla(y-\tilde{y})\|^2 + \nu^{-1}\|p - P_h\|^2 + \nu^{-1}\|\nabla(y-\tilde{y})\|^2 \|\nabla y\|^2 + \nu^{-3}\|\phi_h\|^2\|\nabla y\|^4 + \nu^{-1}\alpha^4 \|\nabla y\|_\infty^2 |y|_2^2 + \nu^{-1}\|\nabla y_h\|^2 \|\nabla(y-\tilde{y})\|^2 \Big).$$

Accordingly, with Gronwall's inequality, the approximation properties and the triangle inequality, we obtain the Lemma. □

5 Numerical Experiments

In this section, a series of numerical experiments are conducted to verify and support the theoretical findings. All computations were carried out using the open-source finite element software FreeFem, as referenced in [24]. For the spatial discretization, the Taylor-Hood finite element pair was employed on regular triangular meshes for the domains specified in each experiment. The control variable was approximated using $P1$ polynomials. To handle the temporal evolution, the Crank-Nicolson (CN) method was utilized for time discretization, while the Newton method was applied to address the nonlinearity in the system. Additionally, the parameter α was always selected to be proportional to the mesh size h, with $\alpha = h$ being used consistently, as indicated in [9].

Initially, a convergence test is conducted to evaluate error rates by comparing the numerical results to a specific analytical solution. The analysis is performed over the time interval $[0, 1]$, with the unit square selected as the spatial domain for the experiments. The following manufactured solutions will be employed for this purpose:

$$y(t,x) = e^{-\nu t} \begin{bmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{bmatrix},$$

In this case, the errors and convergence rates are shown in Table 10. All methods exhibit a consistent reduction in errors as the grid is refined, confirming that the numerical methods are converging as expected. The convergence rates hover around 3 for all methods at finer grids, which suggests second- to third-order accuracy, depending on the method. The differences

Table 10: *The convergence rates and errors for* $\Delta t = 0.0001$ *and* $\nu = 1$ *are presented.*

h	$\|y - y^h\|_{Leray}$	Rate	$\|y - y^h\|_{NS-\alpha}$	Rate	$\|y - y^h\|_{NS-\omega}$	Rate
2^{-1}	0.0460	-	0.0014818	-	0,00148818	-
2^{-2}	0.0073	2.65	2,87621e-4	2.65	2,87622e-4	2.37
2^{-3}	0.0008	3.19	3,25096e-5	3.57	3,25097e-5	3.15
2^{-4}	0.0001	3.01	3,81827e-6	3.32	3,81835e-6	3.09

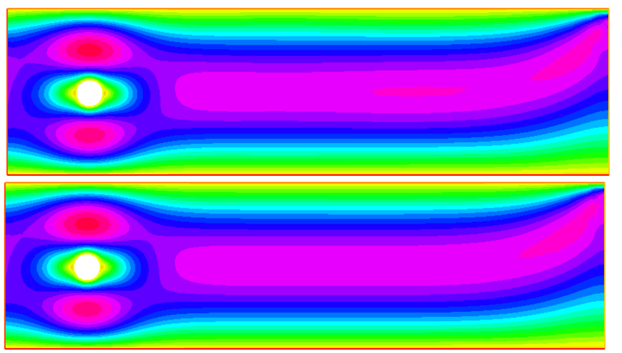


Figure 26: *Velocity profiles for* $Re = 1$.

in the rates between NS-α and NS-ω at finer levels are marginal, indicating similar performance, although NS-α has slightly better convergence rates overall. This detailed convergence analysis demonstrates that the numerical methods (Leray-α, NS-α, and NS-ω) are reliable and consistent in their error reduction as the discretization parameter h becomes smaller.

The Leray-α, NS-α and NS-ω models perform differently in different turbulence environments and each may be better suited for certain conditions due to their unique characteristics. Therefore, the choice of which model is better depends on the type of application and flow conditions. The selection of the appropriate model is important for accurate modeling of the application.

In a separate experiment, the distinctions between general turbulence models and the Stokes solution are analyzed and demonstrated (Fig. 1). The boundary conditions in this specific configuration are described as nonslip around the cylinder submerged in the flow and along the computing domain's horizontal walls. This ensures that the fluid velocity at these surfaces remains zero, reflecting a realistic physical scenario where the fluid adheres to the boundaries. Meanwhile, the boundary conditions at the vertical inlet and outlet are prescribed to follow a parabolic profile in the x-direction, allowing for a smooth and realistic entrance and exit of the fluid. The exact specifications for these parabolic boundary conditions are detailed as follows:

$$y = \begin{pmatrix} 4x_2(1 - x_2) \\ 0 \end{pmatrix}.$$

This equation represents a parabolic boundary condition that defines the velocity profile of the flow at the vertical inlet and outlet. The velocity distribution varies along the x_2 coordinate, describing how the flow behaves at the horizontal direction during entry and exit. The parabolic profile shows that the velocity is highest in the middle and zero at the boundary surfaces, meaning the fluid adheres to the boundaries (i.e., no-slip boundary condition). The term $4x_2(1 - x_2)$ indicates that the fluid reaches its maximum velocity along the centerline and drops to zero at the boundaries. This profile realistically models both the entrance and exit conditions, contributing to more accurate simulation results.

6 Conclusion

This research is poised to provide significant benefits to a diverse audience, including engineers, applied mathematicians, and software developers, by aiding in the creation of more sophisticated algorithms for numerical simulations of turbulent flows. The advancements made through this work can enhance the accuracy and efficiency of simulations, which are critical in understanding and predicting complex fluid behaviors in various contexts.

Moreover, the findings of this study have the potential to contribute substantially to societal welfare across a range of scientific and industrial applications. For instance, in the field of weather forecasting, improved numerical simulations can lead to more accurate predictions of meteorological phenomena, aiding in disaster preparedness and resource management. In terms of energy efficiency, better algorithms can optimize the design of systems that rely on fluid dynamics, such as HVAC systems or renewable energy technologies, thereby reducing energy consumption and environmental impact.

Additionally, in the processing of polymeric materials, such as composite mixing, film blowing, injection molding, and yarn spinning, enhanced simulation tools can facilitate the design and production of materials with superior properties, leading to innovations in manufacturing processes and product quality.

The implications of this research extend to the design of biomedical devices as well, where fluid dynamics plays a crucial role in the development of devices such as drug delivery systems, stents, and artificial organs. Improved simulations can lead to better-performing devices, ultimately enhancing patient outcomes.

Furthermore, this work opens up opportunities for the exploration of new research areas, particularly within the framework of industry-university collaboration. Such partnerships can foster innovation by bringing together academic research and practical industrial applications. As a logical progression, the integration of continuous data assimilation strategies could be considered as a next step. By leveraging real-time data from various applications, researchers can refine these numerical models, making them more responsive and accurate in reflecting real-world conditions.

This multifaceted approach not only enhances the immediate applications of the research but also sets the stage for future advancements that can significantly impact multiple sectors of society.

References

[1] Ŝolín, P. (2005). *Partial differential equations and the finite element method.* John Wiley and Sons.

[2] Brenner, S. C. (2008). *The mathematical theory of finite element methods.* Springer.

[3] Palmer, T. N. (2012). Towards the probabilistic Earth-system simulator: A vision for the future of climate and weather prediction. *Quarterly Journal of the Royal Meteorological Society*, 138(665), 841–861.

[4] Chen, S., Foias, C., Holm, D. D., Olson, E., Titi, E. S., and Wynne, S. (1999). A connection between the Camassa–Holm equations and turbulent flows in channels and pipes. *Physics of Fluids*, 11(8), 2343–2353.

[5] Müller, M., Schirm, S., and Teschner, M. (2004). Interactive blood simulation for virtual surgery based on smoothed particle hydrodynamics. *Technology and Health Care*, 12(1), 25–31.

[6] Cao, Q., Pui, D. Y., and Lipinski, W. (2015). A concept of a novel solar-assisted large-scale cleaning system (SALSCS) for urban air remediation. *Aerosol and Air Quality Research*, 15(1), 1–10.

[7] Cheskidov, A., Holm, D. D., Olson, E., and Titi, E. S. (2005). On a Leray-α model of turbulence. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2055), 629–649.

[8] Layton, W., Manica, C. C., Neda, M., and Rebholz, L. G. (2008). Numerical analysis and computational testing of a high accuracy Leray-deconvolution model of turbulence. *Numerical Methods for Partial Differential Equations: An International Journal*, 24(2), 555–582.

[9] John, V. (2016). Finite element methods for incompressible flow problems. *Cham: Springer International Publishing*, 51, 751–752.

[10] Takhirov, A., and Trenchea, C. (2022). Efficient nonlinear filter stabilization of the Leray-α model. *Journal of Computational Physics*, 471, 111668.

[11] Foias, C., Holm, D. D., & Titi, E. S. (2001). The Navier–Stokes-alpha model of fluid turbulence. Physica D: Nonlinear Phenomena, 152, 505-519.

[12] Foias, C., Holm, D. D., & Titi, E. S. (2002). The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. Journal of Dynamics and Differential Equations, 14, 1-35.

[13] Connors, J. (2010). Convergence analysis and computational testing of the finite element discretization of the Navier–Stokes alpha model. *Numerical Methods for Partial Differential Equations*, 26(6), 1328–1350.

[14] Gutiérrez-Santacreu, J. V., and Rojas-Medar, M. A. (2023). On the approximation of turbulent fluid flows by the Navier–Stokes-α equations on bounded domains. *Physica D: Nonlinear Phenomena*, 448, 133724.

[15] Layton, W., Stanculescu, I., and Trenchea, C. (2011). Theory of the NS-ω model: a complement to the NS-α model. *Communications on Pure and Applied Analysis*, 10(6), pp. 1763-–1777.

[16] Layton, W., Manica, C. C., Neda, M., and Rebholz, L. G. (2010). Numerical analysis and computational comparisons of the NS-alpha and NS-omega regularizations. *Computer Methods in Applied Mechanics and Engineering*, 199(13-16), 916–931.

[17] Temam, R. (1995). *Navier–Stokes equations and nonlinear functional analysis*. Society for Industrial and Applied Mathematics.

[18] Temam, R. (2001). *Navier-Stokes equations: theory and numerical analysis* (Vol. 343). American Mathematical Society.

[19] Layton, W. (2008). *Introduction to the numerical analysis of incompressible viscous flows.* Society for Industrial and Applied Mathematics.

[20] Girault, V., and Raviart, P. A. (1979). *Finite element approximation of the Navier-Stokes equations (Vol. 749).* Berlin: Springer.

[21] Heywood, J., and Rannacher, R. (1986). Finite element approximation of the nonstationary Navier-Stokes equations. *Part II: Stability of Solutions and Error Estimates Uniform in Time, SIAM Journal on Numerical Analysis*, 23, 750–777.

[22] Leray, J. (1934). Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63, 193–248.

[23] Busa, A. (2014). *The Leray-α model of turbulence*. Technical Report, MS Thesis, Free University of Berlin.

[24] Hecht, F. (2012). New development in FreeFem++. *Journal of Numerical Mathematics*, 20(3-4), 251–266.

14. ON CLASSES OF AGGREGATION FUNCTION ON BOUNDED LATTICES

Merve Yeşilyurt **Ümit Ertuğrul**

Abstract

Nullnorms are one of the important classes of aggregation operators. They generalize the notion of triangular norms and triangular conorms. Recently, nullnorms on bounded lattices have been frequently investigated by many esearchers. Construction methods of nullnorms are important since they are also important for ordinal sum construction of nullnorms. In this paper, we present methods to produce nullnorms with the zero element k via the given nullnorm and some other aggregation operators. Then, we demonstrate that our new construction method is also different from the existing construction methods in the literature. Additionally, some illustrative examples are provided.

Keywords. Nullnorm, T-norm, T-conorm, Bounded lattice, Sublattice

ORCID: 0009-0007-7182-6423
e-mail: merveyesilyurt61@hotmail.com

ORCID: 0000-0003-0672-8134
e-mail: uertugrul@ktu.edu.tr

1 Introduction

Aggregation functions are very important operators [13, 22] in the fuzzy set theory and its applications. They were first introduced in [2], as a generalization of t-norms and t-conorms.

Karaçal et al [21] showed they are always exist on every lattice. Also, they determined the smallest nullnorm and the greatest nullnorm. Nullnorms have been also studied from different aspects in the literature [5, 7, 8, 9, 10, 11, 27].

In this paper, we obtain nullnorms from a nullnorm, a t-norm and a t-conorm on subintervals of L under the condition that every element of L is comparable to the elements a and b.

The paper is organized as follows. In Section 2, we recall notions of a bounded lattice and aggregation functions that we will use. Section 3 contains the new construction method via a nullnorm on the subinterval of L.

2 Notations, definitions and a review of previous results

In this chapter, we remind some basic definitons and results.

Definition 2.1. [1] If a lattice L has the top element 1 and the bottom element 0,it is called as a bounded lattice

Definition 2.2. [1] Let L be a bounded lattice. The elements x and y, which satisfy $x \leq y$ or $y \leq x$ are called comparable. Otherwise, they are called incomparable and the notation $x||y$ is used. The set $I_a = \{x \in L \, : \, x||a\}$ is the set of elements which are incomparable elements with a.

Similarly, the set $I_k^{a,b} = \{x \in L \, : \, x||k$ and $x \nparallel a$ and $x \nparallel b\}$ is defined.

Definition 2.3. [1] Let a, b be the elements of a bounded lattice L with $a \leq b$. The $[a, b]$ is a sublattice and given as

$$[a, b] = \{x \in L| \quad a \leq x \leq b\}.$$

Similarly, $(a, b]$, $[a, b)$ and (a, b) are given.

Definition 2.4. [22] If a function T (S) on a bounded lattice L is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0), it is called a triangular norm (triangular conorm).

Example 2.5. The smallest t-norm T_W and the greatest t-norm $T_\wedge$ on bounded lattice L are given respectively as:

$$T_W(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

is the smallest t-norm, and

$$T_{\wedge}(x,y)=x\wedge y$$

is the greatest t-norm on a bounded lattice L.
Similarly,

$$S_{\vee}(x,y)=x\vee y$$

and

$$S_W(x,y)=\begin{cases} y & \text{if } x=0\\ x & \text{if } y=0\\ 1 & \text{otherwise}\end{cases}$$

are the smallest t-conorm and the greatest t-conorm on a bounded lattice L, respectively.

Definition 2.6. [21] The function F on a bounded lattice L is called a nullnorm if it is commutative, associative, non-decreasing in each variable and there is an element $k\in L$ such that for all $x\in[0,k]$ $F(x,0)=x$ and for all $x\in[k,1]$ $F(x,1)=x$.

It is directly obtained from the definition, the element $k\in L$ is an zero (or annihilator element) for F since $F(x,k)=k$ for all $x\in L$.

The set D_k denotes the set $D_k=[0,k]\times[k,1]\cup[k,1]\times[0,k]$ for $k\in L\setminus\{0,1\}$.

Proposition 2.7. [14, 21] Let V be a nullnorm on a bounded lattice L with the zero element $k\in L\setminus\{0,1\}$.
(i) If $(x,y)\in D_k$, $V(x,y)=k$.
(ii) If $(x,y)\in[k,1]^2\cup[k,1]\times I_k\cup I_k\times[k,1]$, $k\leq V(x,y)$.
(iii) If $(x,y)\in[0,k]^2\cup[0,k]\times I_k\cup I_k\times[0,k]$, $V(x,y)\leq k$.
(iv) If $(x,y)\in L\times[k,1]$, $V(x,y)\leq y$.
(v) If $(x,y)\in[k,1]\times L$, $V(x,y)\leq x$.
(vi) If $(x,y)\in[0,k]\times L$, $x\leq V(x,y)$.
(vii) If $(x,y)\in L\times[0,k]$, $y\leq V(x,y)$.
$(viii)$ If $(x,y)\in[0,k]^2$, $x\vee y\leq V(x,y)$.
(ix) If $(x,y)\in[k,1]^2$, $V(x,y)\leq x\wedge y$.
(x) If $(x,y)\in[0,k]\times I_k\cup I_k\times[0,k]\cup I_k\times I_k$, $(x\wedge k)\vee(y\wedge k)\leq V(x,y)$.
(xi) If $(x,y)\in[k,1]\times I_k\cup I_k\times[k,1]\cup I_k\times I_k$, $V(x,y)\leq(x\vee k)\wedge(y\vee k)$.

3 Construction of nullnorms on bounded lattices

In this chapter, after reminding the methods to obtain nullnorms in the literature, we propose an extension method for nullnorms in Theorems 5 on a bounded lattice L via the existence of a nullnorm V on $[a,b]$ of L, where $x \not\parallel a$ and $x \not\parallel b$ for all $x \in L$. Some illustrative examples are provided to clarify the method.

Theorem 3.1. [21] Let L be a bounded lattice, $k \in L \backslash \{0,1\}$, S be a t-conorm on $[0,k]$ and T be a t-norm on $[k,1]$. Then, the functions $V_k^S, V_k^T : L^2 \to L$ defined as follows

$$V_k^S(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,k]^2, \\ k & (x,y) \in [k,1)^2 \cup [k,1] \times I_k \cup I_k \times [k,1] \cup D_k, \\ S(x \wedge k, y \wedge k) & (x,y) \in [0,k] \times I_k \cup I_k \times [0,k] \cup I_k \times I_k, \\ x \wedge y & \text{otherwise} \end{cases} \tag{3.2}$$

and

$$V_k^T(x,y) = \begin{cases} T(x,y) & (x,y) \in [k,1]^2, \\ k & (x,y) \in [0,k)^2 \cup [0,k] \times I_k \cup I_k \times [0,k] \cup D_k, \\ T(x \vee k, y \vee k) & (x,y) \in [k,1] \times I_k \cup I_k \times [k,1] \cup I_k \times I_k, \\ x \vee y & \text{otherwise} \end{cases} \tag{3.3}$$

are nullnorms on L with zero element k.

Theorem 3.4. [7] Let L be a bounded lattice, $a, k \in L \setminus \{0,1\}$ and $k \in [0,a]$ such that $x \not\parallel k$ for all $x \in [0,a]$. If $x \geq a$ for all $x \in L \setminus [0,a]$, V^* is a nullnorm on $[0,a]$ with the zero element k and T is a t-norm on $[a,1]$, then the following operation $V_1 : L^2 \to L$ is a nullnorm with the zero element k, where

$$V_1(x,y) = \begin{cases} V^*(x,y) & (x,y) \in [0,a]^2, \\ k & (x,y) \in [0,k] \times [a,1] \cup [a,1] \times [0,k], \\ T(x,y) & (x,y) \in [a,1]^2, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3.5}$$

Theorem 3.6. [7] Let L be a bounded lattice, $a, k \in L \setminus \{0,1\}$ and $k \in [a,1]$ such that $x \not\parallel k$ for all $x \in [a,1]$. If $x \leq a$ for all $x \in L \setminus [a,1]$, V_* is a nullnorm on $[a,1]$ with the zero element k and S is a t-conorm on $[0,a]$, then the following operation $V_2 : L^2 \to L$ is a nullnorm with the zero element k, where

$$V_2(x,y) = \begin{cases} V_*(x,y) & (x,y) \in [a,1]^2, \\ k & (x,y) \in [k,1] \times [0,a] \cup [0,a] \times [k,1], \\ S(x,y) & (x,y) \in [0,a]^2, \\ x \vee y & \text{otherwise.} \end{cases} \tag{3.7}$$

In the following theorem, based on a nullnorm V on $[a,b]$ on a subinterval of the bounded lattice L, where $I_a = I_b = \emptyset$, we give a method to produce nullnorms on a bounded lattice L.

Theorem 3.8. Let L be a bounded lattice, $a, b, k \in L\setminus\{0, 1\}$ with $a < k < b$ such thet $I_a = I_b = \emptyset$, S be t-conorm on $[0, a]$, T be a t-norm $[b, 1]$, V be a nullnorm on $[a, b]$ with the zero element k. Then the following function $F : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ T(x,y) & (x,y) \in [b,1]^2, \\ V(x,y) & (x,y) \in [a,b]^2, \\ V(a, y \wedge k) & (x,y) \in (0,a) \times I_k^{a,b}, \\ V(x \wedge k, a) & (x,y) \in I_k^{a,b} \times (0,a), \\ V(x \wedge k, y \wedge k) & (x,y) \in ([a,k) \cup I_k^{a,b}) \times I_k^{a,b} \cup I_k^{a,b} \times [a,k), \\ x \vee y & (x,y) \in [0,a) \times [a,k) \cup [a,k) \times [0,a), \\ x \wedge y & (x,y) \in (k,b] \times (b,1] \cup (b,1] \times (k,b], \\ k & \text{otherwise.} \end{cases} \tag{3.9}$$

Proof. (i) *Monotonicity*: Let us show that for every elements $x, y \in L$ with $x \leq y$, $F(x, z) \leq F(y, z)$ for all $z \in L$. If x and y are both elements of $[0, a)$ or $[a, k]$ or $(k, b]$ or $(b, 1]$ or $I_k^{a,b}$, $F(x, z) \leq F(y, z)$ is always satisfied for all $z \in L$ since $x \leq y$. It is clear that $F(x, z) = k = F(y, z)$, when $z = k$. The proof is then split into all the remain possible cases as follows.

1. Let $x \in [0, a)$.

1.1. $y \in [a, k)$,

1.1.1. If $z \in [0, a)$, then $F(x, z) = S(x, z) \leq a \leq y = y \vee z = F(y, z)$.

1.1.2. If $z \in [a, k)$, then $F(x, z) = x \vee z \leq y \vee z \leq V(y, z) = F(y, z)$.

1.1.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, z) = k = F(y, z)$.

1.1.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(a, z \wedge k) \leq V(y \wedge k, z \wedge k) = F(y, z)$.

1.2. $y = k$,

1.2.1. If $z \in [0, a)$, then $F(x, z) = S(x, z) \leq a \leq k = F(y, z)$.

1.2.2. If $z \in [a, k)$, then $F(x, z) = x \vee z = z \leq k = F(y, z)$.

1.2.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, z) = k = F(y, z)$.

1.2.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(a, z \wedge k) \leq V(a, k) = k = F(y, z)$.

1.3. $y \in (k, b]$,

1.3.1. If $z \in [0, a)$, then $F(x, z) = S(x, z) \leq a \leq k = F(y, z)$.

1.3.2. If $z \in [a, k)$, then $F(x, z) = x \vee z = z \leq k = V(y, z) = F(y, z)$.

1.3.3. If $z \in (k, b]$, then $F(x, z) = k = V(k, z) \leq V(y, z) = F(y, z)$.

1.3.4. If $z \in (b, 1]$, then $F(x, z) = k \leq y = y \wedge z = F(y, z)$.

1.3.5. If $z \in I_k^{a,b}$, then $F(x, z) = V(a, z \wedge k) \leq V(a, k) = k = F(y, z)$.

1.4. $y \in (b, 1]$,

1.4.1. If $z \in [0, a)$, then $F(x, z) = S(x, z) \leq a \leq k = F(y, z)$.

1.4.2. If $z \in [a, k)$, then $F(x, z) = x \vee z = z \leq k = F(y, z)$.

1.4.3. If $z \in (k, b]$, then $F(x, z) = k \leq z = y \wedge z = F(y, z)$.

1.4.4. If $z \in (b, 1]$, then $F(x, z) = k \leq b \leq T(y, z) = F(y, z)$.

1.4.5. If $z \in I_k^{a,b}$, then $F(x, z) = V(a, z \wedge k) \leq V(a, k) = k = F(y, z)$.

1.5. $y \in I_k^{a,b}$,

1.5.1. If $z \in [0, a)$, then $F(x, z) = S(x, z) \leq a \leq V(y \wedge k, a) = F(y, z)$.

1.5.2. If $z \in [a, k)$, then $F(x, z) = x \vee z = z = V(a, z) \leq V(y \wedge k, z \wedge k) = F(y, z)$.

1.5.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, z) = k = F(y, z)$.

1.5.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(a, z \wedge k) \leq V(y \wedge k, z \wedge k) = F(y, z)$.

2. Let $x \in [a, k)$.

2.1. $y = k$,

2.1.1. If $z \in [0, a)$, then $F(x, z) = x \vee z = x \leq k = F(y, z)$.

2.1.2. If $z \in [a, k)$, then $F(x, z) = V(x, z) \leq V(k, z) = k = F(y, z)$.

2.1.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, z) = k = F(y, z)$.

2.1.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \leq V(k, z \wedge k) = k = F(y, z)$.

2.2. $y \in (k, b]$,

2.2.1. If $z \in [0, a)$, then $F(x, z) = x \vee z = x \leq k = F(y, z)$.

2.2.2. If $z \in [a, k) \cup (k, b]$, then $F(x, z) = V(x, z) \leq V(y, z) = F(y, z)$.

2.2.3. If $z \in (b, 1]$, then $F(x, z) = k \leq y = y \wedge z = F(y, z)$.

2.2.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \leq V(k, z \wedge k) = k = F(y, z)$.

2.3. $y \in (b, 1]$,

2.3.1. If $z \in [0, a)$, then $F(x, z) = x \vee z = x \leq k = F(y, z)$.

2.3.2. If $z \in [a, k)$, then $F(x, z) = V(x, z) \leq V(k, z) = k = F(y, z)$.

2.3.3. If $z \in (k, b]$, then $F(x, z) = V(x, z) = k \leq z = y \wedge z = F(y, z)$.

2.3.4. If $z \in (b, 1]$, then $F(x, z) = V(x, z) = k \leq b \leq T(y, z) = F(y, z)$.

2.3.5. If $z \in I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \leq V(k, z \wedge k) = k = F(y, z)$.

2.4. $y \in I_k^{a,b}$,

2.4.1. If $z \in [0, a)$, then $F(x, z) = x \vee z = x \leq V(y \wedge k, a) = y \wedge k = F(y, z)$.

2.4.2. If $z \in [a, k)$, then $F(x, z) = V(x, z) \le V(y \wedge k, z \wedge k) = F(y, z)$.

2.4.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, z) = k = F(y, z)$.

2.4.4. If $z \in I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \le V(x \wedge k, z \wedge k) = F(y, z)$.

3. Let $x \in (k, b]$.

3.1. $y \in (b, 1]$,

3.1.1. If $z \in [0, a) \cup [a, k)$, then $F(x, z) = k = F(y, z)$.

3.1.2. If $z \in (k, b]$, then $F(x, z) = V(x, z) \le V(b, z) = z = y \wedge z = F(y, z)$.

3.1.3. If $z \in (b, 1]$, then $F(x, z) = x \wedge z = x \le b \le T(y, z) = F(y, z)$.

3.1.4. If $z \in I_k^{a,b}$, then $F(x, z) = k = F(y, z)$.

4. Let $x \in I_k^{a,b}$.

4.1. $y \in (k, b]$,

4.1.1. If $z \in [0, a)$, then $F(x, z) = V(x \wedge k, a) \le V(y \wedge k, k) = k = F(y, z)$.

4.1.2. If $z \in [a, k) \cup I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \le V(y \wedge k, k) = k = F(y, z)$.

4.1.3. If $z \in (k, b]$, then $F(x, z) = k = V(k, z) \le V(y, z) = F(y, z)$.

4.1.4. If $z \in (b, 1]$, then $F(x, z) = k \le y = y \wedge z = F(y, z)$.

4.2. $y \in (b, 1]$,

4.2.1. If $z \in [0, a)$, then $F(x, z) = V(x \wedge k, a) \le V(y \wedge k, k) = k = F(y, z)$.

4.2.2. If $z \in [a, k) \cup I_k^{a,b}$, then $F(x, z) = V(x \wedge k, z \wedge k) \le V(y \wedge k, k) = k = F(y, z)$.

4.2.3. If $z \in (k, b]$, then $F(x, z) = k \le z = y \wedge z = F(y, z)$.

4.2.4. If $z \in (b, 1]$, then $F(x, z) = k \le b \le T(y, z) = F(y, z)$.

(ii) *Associativity*: We demonstrate that $F(x, F(y, z)) = F(F(x, y), z)$ for all $x, y, z \in L$. If one of the elements x, y and z is equal to k, it is clear that the equality is always satisfied. Again, the proof is split into all remain possible cases by considering the relationships between the elements x, y, z, a, b and k as follows.

1. Let $x \in [0, a)$.

1.1. $y \in [0, a)$,

1.1.1. If $z \in [0, a)$, then $F(x, F(y, z)) = F(x, S(y, z)) = S(x, S(y, z)) = S(S(x, y), z) = F(S(x, y), z) = F(F(x, y), z)$. 1.1.2. If $z \in [a, k)$, then $F(x, F(y, z)) = F(x, y \vee z) = F(x, z) = x \vee z = z = S(x, y) \vee z = F(S(x, y), z) = F(F(x, y), z)$.

1.1.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, F(y, z)) = F(x, k) = k = F(S(x, y), z) = F(F(x, y), z)$.

1.1.4. If $z \in I_k^{a,b}$, then $F(x, F(y,z)) = F(x, V(a, z \wedge k)) = F(x, z \wedge k) = x \vee (z \wedge k) = z \wedge k = V(a, z \wedge k) = F(S(x,y), z) = F(F(x,y), z)$.

1.2. $y \in [a,k)$,

1.2.1. If $z \in [0,a)$, then $F(x, F(y,z)) = F(x, y \vee z) = F(x,y) = x \vee y = y = y \vee z = F(y,z) = F(x \vee y, z) = F(F(x,y), z)$.

1.2.2. If $z \in [a,k)$, then $F(x, F(y,z)) = F(x, V(y,z)) = x \vee V(y,z) = V(y,z) = F(y,z) = F(x \vee y, z) = F(F(x,y), z)$.

1.2.3. If $z \in (k,b]$, then $F(x, F(y,z)) = F(x, V(y,z)) = F(x,k) = k = F(y,z) = F(x \vee y, z) = F(F(x,y), z)$.

1.2.4. If $z \in (b,1]$, then $F(x, F(y,z)) = F(x,k) = k = F(y,z) = F(x \vee y, z) = F(F(x,y), z)$.

1.2.5. If $z \in I_k^{a,b}$, then $F(x, F(y,z)) = F(x, V(y \wedge k, z \wedge k)) = x \vee V(y \wedge k, z \wedge k) = V(y \wedge k, z \wedge k) = F(y,z) = F(x \vee y, z) = F(F(x,y), z)$.

1.3. $y \in (k,b]$,

1.3.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x, F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y), z)$.

1.3.2. If $z \in (k,b]$, then $F(x, F(y,z)) = F(x, V(y,z)) = k = F(k,z) = F(F(x,y), z)$.

1.3.3. If $z \in (b,1]$, then $F(x, F(y,z)) = F(x, y \wedge z) = F(x,y) = k = F(k,z) = F(F(x,y), z)$.

1.4. $y \in (b,1]$,

1.4.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x, F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y), z)$.

1.4.2. If $z \in (k,b]$, then $F(x, F(y,z)) = F(x, y \wedge z) = F(x,z) = k = F(k,z) = F(F(x,y), z)$.

1.4.3. If $z \in (b,1]$, then $F(x, F(y,z)) = F(x, T(y,z)) = k = F(k,z) = F(F(x,y), z)$.

1.5. $y \in I_k^{a,b}$,

1.5.1. If $z \in [0,a)$, then $F(x, F(y,z)) = F(x, V(y \wedge k, a)) = x \vee V(y \wedge k, a) = V(y \wedge k, a) = V(a, y \wedge k) \vee z = F(V(a, y \wedge k), z) = F(F(x,y), z)$.

1.5.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F(x, F(y,z)) = F(x, V(y \wedge k, z \wedge k)) = x \vee V(y \wedge k, z \wedge k) = V(y \wedge k, z \wedge k) = F(y \wedge k, z) = F(V(a, y \wedge k), z) = F(F(x,y), z)$.

1.5.3. If $z \in (k,b] \cup (b,1]$, then $F(x, F(y,z)) = F(x,k) = k = F(y \wedge k, z) = F(V(a, y \wedge k), z) = F(F(x,y), z)$.

2. Let $x \in [a,k)$.

2.1. $y \in [0,a)$,

2.1.1. If $z \in [0,a)$, then $F(x, F(y,z)) = F(x, S(y,z)) = x \vee S(y,z) = x = x \vee z = F(x,z) = F(x \vee y, z) = F(F(x,y), z)$.

2.1.2. If $z \in [a,k)$, then $F(x, F(y,z)) = F(x, y \vee z) = F(x,z) = F(x \vee y, z) = F(F(x,y), z)$.

2.1.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(x,z) = F(x \vee y, z) = F(F(x,y),z)$.

2.1.4. If $z \in I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(a,z \wedge k)) = F(x,z \wedge k) = V(x,z \wedge k) = V(x \wedge k, z \wedge k) = F(x,z) = F(x \vee y, z) = F(F(x,y),z)$.

2.2. $y \in [a,k)$,

2.2.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = V(x,y) = V(x,y) \vee z = F(V(x,y),z) = F(F(x,y),z)$.

2.2.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,V(y,z)) = V(x,V(y,z)) = V(V(x,y),z) = F(V(x,y),z) = F(F(x,y),z)$.

2.2.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,V(y,z)) = F(x,k) = k = F(V(x,y),z) = F(F(x,y),z)$.

2.2.4. If $z \in I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(y \wedge k, z \wedge k)) = V(x,V(y \wedge k, z \wedge k)) = V(V(x,y), z \wedge k) = F(V(x,y),z) = F(F(x,y),z)$.

2.3. $y \in (k,b]$,

2.3.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

2.3.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,V(y,z)) = V(x,V(y,z)) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

2.3.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = V(x,y) = k = F(k,z) = F(V(x,y),z) = F(F(x,y),z)$.

2.4. $y \in (b,1]$,

2.4.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

2.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = V(x,z) = k = F(k,z) = F(F(x,y),z)$.

2.4.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x,T(y,z)) = k = F(k,z) = F(F(x,y),z)$.

2.5. $y \in I_k^{a,b}$,

2.5.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,V(y \wedge k, a)) = F(x,y \wedge k) = V(x,y \wedge k) = V(x \wedge k, y \wedge k) \vee z = F(V(x \wedge k, y \wedge k),z) = F(F(x,y),z)$.

2.5.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(y \wedge k, z \wedge k)) = V(x,V(y \wedge k, z \wedge k)) = V(V(x \wedge k, y \wedge k), z \wedge k) = F(V(x \wedge k, y \wedge k),z) = F(F(x,y),z)$.

2.5.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge k, y \wedge k),z) = F(F(x,y),z)$.

3. Let $x \in (k,b]$.

3.1. $y \in [0,a)$,

3.1.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = k = F(k,z) = F(F(x,y),z)$.

3.1.2. If $z \in [a, k)$, then $F(x, F(y, z)) = F(x, y \vee z) = F(x, z) = V(x, z) = k = F(k, z) = F(F(x, y), z)$.

3.1.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, F(y, z)) = F(x, k) = k = F(k, z) = F(F(x, y), z)$.

3.1.4. If $z \in I_k^{a,b}$, then $F(x, F(y, z)) = F(x, V(a, z \wedge k)) = F(x, z \wedge k) = k = F(k, z) = F(F(x, y), z)$.

3.2. $y \in [a, k)$,

3.2.1. If $z \in [0, a)$, then $F(x, F(y, z)) = F(x, y \vee z) = F(x, y) = V(x, y) = k = F(k, z) = F(V(x, y), z) = F(F(x, y), z)$.

3.2.2. If $z \in [a, k)$, then $F(x, F(y, z)) = F(x, V(y, z)) = V(x, V(y, z)) = k = F(k, z) = F(V(x, y), z) = F(F(x, y), z)$.

3.2.3. If $z \in (k, b]$, then $F(x, F(y, z)) = F(x, V(y, z)) = F(x, k) = k = F(k, z) = F(V(x, y), z) = F(F(x, y), z)$.

3.2.4. If $z \in (b, 1]$, then $F(x, F(y, z)) = F(x, k) = k = F(k, z) = F(V(x, y), z) = F(F(x, y), z)$.

3.2.5. If $z \in I_k^{a,b}$, then $F(x, F(y, z)) = F(x, V(y \wedge k, z \wedge k)) = k = F(k, z) = F(V(x, y), z) = F(F(x, y), z)$.

3.3. $y \in (k, b]$,

3.3.1. If $z \in [0, a) \cup [a, k) \cup I_k^{a,b}$, then $F(x, F(y, z)) = F(x, k) = k = F(V(x, y), z) = F(F(x, y), z)$.

3.3.2. If $z \in (k, b]$, then $F(x, F(y, z)) = F(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = F(V(x, y), z) = F(F(x, y), z)$.

3.3.3. If $z \in (b, 1]$, then $F(x, F(y, z)) = F(x, y \wedge z) = F(x, y) = V(x, y) = V(x, y) \wedge z = F(V(x, y), z) = F(F(x, y), z)$.

3.4. $y \in (b, 1]$,

3.4.1. If $z \in [0, a) \cup [a, k) \cup I_k^{a,b}$, then $F(x, F(y, z)) = F(x, k) = k = F(x, z) = F(x \wedge y, z) = F(F(x, y), z)$.

3.4.2. If $z \in (k, b]$, then $F(x, F(y, z)) = F(x, y \wedge z) = F(x, z) = F(x \wedge y, z) = F(F(x, y), z)$.

3.4.3. If $z \in (b, 1]$, then $F(x, F(y, z)) = F(x, T(y, z)) = x \wedge T(y, z) = x = x \wedge z = F(x, z) = F(x \wedge y, z) = F(F(x, y), z)$.

3.5. $y \in I_k^{a,b}$,

3.5.1. If $z \in [0, a)$, then $F(x, F(y, z)) = F(x, V(y \wedge k, a)) = V(x, y \wedge k) = k = F(k, z) = F(F(x, y), z)$.

3.5.2. If $z \in [a, k) \cup I_k^{a,b}$, then $F(x, F(y, z)) = F(x, V(y \wedge k, z \wedge k)) = k = F(k, z) = F(F(x, y), z)$.

3.5.3. If $z \in (k, b] \cup (b, 1]$, then $F(x, F(y, z)) = F(x, k) = k = F(k, z) = F(F(x, y), z)$.

4. Let $x \in (b, 1]$.

4.1. $y \in [0, a)$,

4.1.1. If $z \in [0, a)$, then $F(x, F(y, z)) = F(x, S(y, z)) = k = F(k, z) = F(F(x, y), z)$.

4.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = k = F(k,z) = F(F(x,y),z)$.

4.1.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

4.1.4. If $z \in I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(a,z \wedge k)) = k = F(k,z) = F(F(x,y),z)$.

4.2. $y \in [a,k)$,

4.2.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = k = F(k,z) = F(F(x,y),z)$.

4.2.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,V(y,z)) = k = F(k,z) = F(F(x,y),z)$.

4.2.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

4.2.4. If $z \in I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(y \wedge k, z \wedge k)) = k = F(k,z) = F(F(x,y),z)$.

4.3. $y \in (k,b]$,

4.3.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(y,z) = F(x \wedge y, z) = F(F(x,y),z)$.

4.3.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,V(y,z)) = x \wedge V(y,z) = V(y,z) = F(y,z) = F(x \wedge y,z) = F(F(x,y),z)$.

4.3.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,y) = x \wedge y = y = y \wedge z = F(y,z) = F(x \wedge y,z) = F(F(x,y),z)$.

4.4. $y \in (b,1]$,

4.4.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(T(x,y),z) = F(F(x,y),z)$.

4.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x,y \wedge z) = F(x,z) = x \wedge z = z = T(x,y) \wedge z = F(T(x,y),z) = F(F(x,y),z)$.

4.4.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x,T(y,z)) = T(x,T(y,z)) = T(T(x,y),z) = F(T(x,y),z) = F(F(x,y),z)$.

4.5. $y \in I_k^{a,b}$,

4.5.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,V(y \wedge k,a)) = V(x,y \wedge k) = k = F(k,z) = F(F(x,y),z)$.

4.5.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,V(y \wedge k, z \wedge k)) = k = F(k,z) = F(F(x,y),z)$.

4.5.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y),z)$.

5. Let $x \in I_k^{a,b}$.

5.1. $y \in [0,a)$,

5.1.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,S(y,z)) = V(x \wedge k,a) = x \wedge k = (x \wedge k) \vee z = F(x \wedge k,z) = F(V(x \wedge k,a),z) = F(F(x,y),z)$.

5.1.2. If $z \in [a,k)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,z) = V(x \wedge k, z \wedge k) = F(x \wedge k, z) = F(V(x \wedge k, a), z) = F(F(x,y), z)$.

5.1.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(x \wedge k, z) = F(V(x \wedge k, a), z) = F(F(x,y), z)$.

5.1.4. If $z \in I_k^{a,b}$, then $F(x,F(y,z)) = F(x, V(a, z \wedge k)) = F(x, z \wedge k) V(x \wedge k, z \wedge k) = F(x \wedge k, z) = F(V(x \wedge k, a), z) = F(F(x,y), z)$.

5.2. $y \in [a,k)$,

5.2.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x,y \vee z) = F(x,y) = V(x \wedge k, y \wedge k) = V(x \wedge k, y \wedge k) \vee z = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

5.2.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x, V(y \wedge k, z \wedge k)) = V(x \wedge k, V(y \wedge k, z \wedge k)) = V(V(x \wedge k, y \wedge k), z \wedge k) = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

5.2.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

5.3. $y \in (k,b]$,

5.3.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y), z)$.

5.3.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x, V(y,z)) = k = F(k,z) = F(F(x,y), z)$.

5.3.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x, y \wedge z) = F(x,y) = k = F(k,z) = F(F(x,y), z)$.

5.4. $y \in (b,1]$,

5.4.1. If $z \in [0,a) \cup [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x,k) = k = F(k,z) = F(F(x,y), z)$.

5.4.2. If $z \in (k,b]$, then $F(x,F(y,z)) = F(x, y \wedge z) = F(x,z) = k = F(k,z) = F(F(x,y), z)$.

5.4.3. If $z \in (b,1]$, then $F(x,F(y,z)) = F(x, T(y,z)) = k = F(k,z) = F(F(x,y), z)$.

5.5. $y \in I_k^{a,b}$,

5.5.1. If $z \in [0,a)$, then $F(x,F(y,z)) = F(x, V(y \wedge k, a)) = F(x, y \wedge k) = V(x \wedge k, y \wedge k) = V(x \wedge k, y \wedge k) \vee z = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

5.5.2. If $z \in [a,k) \cup I_k^{a,b}$, then $F(x,F(y,z)) = F(x, V(y \wedge k, z \wedge k)) = V(x \wedge k, V(y \wedge k, z \wedge k)) = V(V(x \wedge k, y \wedge k), z \wedge k) = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

5.5.3. If $z \in (k,b] \cup (b,1]$, then $F(x,F(y,z)) = F(x,k) = k = F(V(x \wedge k, y \wedge k), z) = F(F(x,y), z)$.

We have $F(x,0) = x \vee 0 = x$ for all $x \in [0,k]$ and $F(t,1) = t \wedge 1 = t$ for all $t \in [k,1]$. The commutativity of F is obvious from the definition of F. Therefore, F is a nullnorm on L with the zero element k.

The structure of the nullnorm F given in formula (5) can be summarized Figure 1.

$I_k^{a,b}$	$V(a, y \wedge k)$	$V(x \wedge k, y \wedge k)$	k	k	$V(x \wedge k, y \wedge k)$
1	k	k	$x \wedge y$	$T(x, y)$	k
b	k	$V(x, y)$	$V(x, y)$	$x \wedge y$	k
k	$x \vee y$	$V(x, y)$	$V(x, y)$	k	$V(x \wedge k, y \wedge k)$
a	$S(x, y)$	$x \vee y$	k	k	$V(x \wedge k, a)$
0	a	k	b	1	$I_k^{a,b}$

Figure 27: *The structure of the nullnorm F*

$\square$

Remark 3.10. Note that in Theorem 3, $I_a = I_k = \emptyset$. If $a = 0$ and $I_k^{a,b} = \emptyset$ in the formula (5), the formula (5) coincides the formula (3). (Note that the element b in the formula (5) corresponds the element a in formula (3)). Therefore, Theorem 5 is more general than Theorem 3.

Similarly, in Theorem 4, $I_a = I_k = \emptyset$. If $b = 1$ and $I_k^{a,b} = \emptyset$ in the formula (5), the formula (5) coincides the formula (4). Therefore, Theorem 5 is more general than Theorem 4.

Example 3.11. Consider the bounded lattice $(L_1 = \{0, a, b, c, d, k, 1\}, \leq, 0, 1)$ characterized in Figure 2 and the nullnorm V on $[a, b]$ with zero element k as in Table 1.

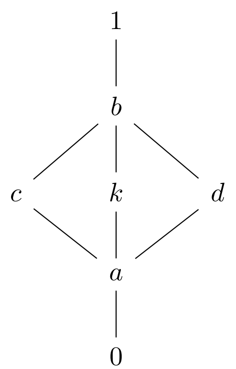


Figure 28: Lattice diagram of L_1.

V	a	b	c	d	k
a	a	k	a	a	k
b	k	b	k	k	k
c	a	k	a	a	k
d	a	k	a	a	k
k	k	k	k	k	k

Table 11: *The nullnorm V on* $[a, b]$.

Considering the method in Theorem 5 and putting the nullnorm V on $[a, b]$ as in Table 1, the corresponding nullnorm F on L_1 is obtained as in Table 2 based on Theorem 5.

F	0	a	b	c	d	k	1
0	0	a	k	a	a	k	k
a	a	a	k	a	a	k	k
b	k	k	b	k	k	k	b
c	a	a	k	a	a	k	k
d	a	a	k	a	k	k	k
k	k	k	k	a	k	k	k
1	k	k	b	k	k	k	1

Table 12: *The nullnorm F obtained by the formula* (5) *in Theorem* 5.

If we put $T_\wedge = T$ and $S_\vee = S$ in Theorem 5 in the formula (5), the following construction method is obtained. The following method is an extension method of a nullnorm V on a subinterval $[a,b]$ of a given bounded lattice L to the whole lattice.

Corollary 3.12. Let L be a bounded lattice, $a,b,k \in L \setminus \{0,1\}$ with $a < k < b$, S be t-conorm on $[0,a]$, T be a t-norm $[b,1]$, V be a nullnorm on $[a,b]$ with the zero element k and comparable with a and b all elements of L. Then the following function $F_1 : L^2 \to L$ is a nullnorm with the zero element k on L, where

$$F_1(x,y) = \begin{cases} V(x,y) & (x,y) \in [a,b]^2, \\ V(a, y \wedge k) & (x,y) \in (0,a) \times I_k^{a,b}, \\ V(x \wedge k, a) & (x,y) \in I_k^{a,b} \times (0,a), \\ V(x \wedge k, y \wedge k) & (x,y) \in ([a,k) \cup I_k^{a,b}) \times I_k^{a,b} \cup I_k^{a,b} \times [a,k), \\ x \vee y & (x,y) \in [0,a]^2 \cup [0,a) \times [a,k) \cup [a,k) \times [0,a), \\ x \wedge y & (x,y) \in [b,1]^2 \cup (k,b] \times (b,1] \cup (b,1] \times (k,b], \\ k & \text{otherwise.} \end{cases} \tag{3.13}$$

In the following example, we show that Theorem 5 may not produce a nullnorm on a bounded lattice L if $I_a \neq \emptyset$ or $I_b \neq \emptyset$.

Example 3.14. Consider the bounded lattice $(L_2 = \{0,a,b,k,p,s,t,1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 3, the nullnorm V on $[a,b]$.

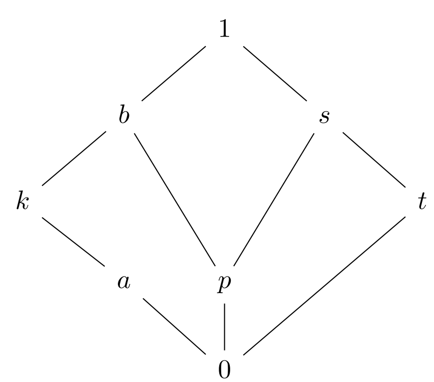


Figure 29: Lattice diagram of L_2.

Therefore, it is clear that $I_b \neq \emptyset$ $(s||b)$ since $F(a, F(b,s)) \neq F(F(a,b),s)$, Theorem 5 does not work if $I_a \neq \emptyset$ or $I_b \neq \emptyset$.

4 Concluding remarks

In this study, we proposed construction methods for nullnormses. There are some known construction methods on bounded lattices using the existence of some spesific aggregation functions, like t-norms and t-conorms in the literature but according to our best knowledge, there is only one study to construct a nullnorm from a given nullnorm and a t-norm (a t-conorm) on sublattices. In this paper, we investigated how to extend a given nullnorm on a subinterval to the whole lattice L. This construction method can be seen asan extension method for nullnorms on a bounded lattice by exploiting the nullnorm V on a sublattice $[a, b]$, when $I_a = I_b = \emptyset$.

5 Acknowledgements

This study was supported by Scientific and Technological Research Council of Turkey (TÜBİTAK) under the Grant Number 122F472. The authors thank to TÜBİTAK for their supports. This study was supported by Office of Scientific Research Projects of Karadeniz Technical University. Project Number: FDK-2022-10396.

References

[1] Birkhoff, G. (1940). Lattice theory (Vol. 25). American Mathematical Soc..

[2] Calvo, T., & De Baets, B., & Fodor, J. (2001). The functional equations of Frank and Alsina for uninorms and nullnorms. Fuzzy Sets and Systems, 120(3), 385-394.

[3] Castro, J. L. (1995). Fuzzy logic controllers are universal approximators. IEEE transactions on systems, man, and cybernetics, 25(4), 629-635.

[4] Combs, W. E., & Andrews, J. E. (1998). Combinatorial rule explosion eliminated by a fuzzy rule configuration. IEEE Transactions on fuzzy Systems, 6(1), 1-11.

[5] Çaylı, G. D., & Karaçal, F. (2018). Idempotent nullnorms on bounded lattices. Information Sciences, 425, 154-163.

[6] Çaylı, G. D., & Karaçal, F. (2018). Some remarks on idempotent nullnorms on bounded lattices. In Aggregation Functions in Theory and in Practice (pp. 31-39). Springer International Publishing.

[7] Çaylı, G. D., & Karaçal, F. (2018). A survey on nullnorms on bounded lattices. In Advances in Fuzzy Logic and Technology 2017: Proceedings of: EUSFLAT-2017–The 10th Conference of the European Society for Fuzzy Logic and Technology, September 11–15, 2017, Warsaw, Poland IWIFSGN'2017–The Sixteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, September 13–15, 2017, Warsaw, Poland, Volume 1 10 (pp. 431-442). Springer International Publishing.

[8] Çaylı, G. D. (2020). Nullnorms on bounded lattices derived from t-norms and t-conorms. Information Sciences, 512, 1134-1154.

[9] Çaylı, G. D. (2020). Construction methods for idempotent nullnorms on bounded lattices. Applied Mathematics and Computation, 366, 124746.

[10] Çaylı, G. D. (2020). Some results about nullnorms on bounded lattices. Fuzzy Sets and Systems, 386, 105-131.

[11] Çaylı, G. D. (2021). Construction of nullnorms on some special classes of bounded lattices. International Journal of Approximate Reasoning, 134, 111-128.

[12] Dan, Y., Hu, B. Q., & De Baets, B. (2022). Nullnorms on bounded lattices constructed by means of closure and interior operators. Fuzzy Sets and Systems, 439, 142-156.

[13] Dubois, D., & Prade, H. (Eds.). (2012). Fundamentals of fuzzy sets (Vol. 7). Springer Science & Business Media.

[14] Drygaś, P. (2004). Isotonic operations with zero element in bounded lattices. In: Atanassov, K., Hryniewicz, O., Kacprzyk, J. (eds.) Soft Computing Foundations and Theoretical Aspect, EXIT Warszawa, 181–190.

[15] Dvorak, A., & Novak, V. (2000). On the extraction of linguistic knowledge in data bases using fuzzy logic, in Flexible Query Answering Systems: Recent Advances, H. L. Larsen, Ed. Heidelberg, Germany: Physika-Verlag, 445–454.

[16] Ertuğrul, Ü. (2018). Construction of nullnorms on bounded lattices and an equivalence relation on nullnorms, Fuzzy sets and systems, 334, 94-109.

[17] Ertuğrul, Ü.,& Yeşilyurt, M. (2020). Ordinal sums of triangular norms on bounded lattices, Inf. Sci. 517, 198–216.

[18] Fodor, J. C., & Roubens, M. (1994). Fuzzy preference modelling and multicriteria decision support, in Theory and Library, Series D: System Theory, Knowledge Engineering and Problem Solving, Dordrecht, The Netherlands: Kluwer Academic.

[19] Godo, L., & Sierra, C. (1988). A new approach to connective generation in the framework of expert systems using fuzzy logic, In: Proc. XVIIIth ISMVL, Palma de Mallorca, Spain, 157–162.

[20] Grabisch, M. (2009). Marichal, J.-L., Mesiar, R., and Pap, E. Aggregation Functions, Cambridge University Press.

[21] Karaçal, F., & İnce, M. A., & Mesiar, R. (2015). Nullnorms on bounded lattices, Information Sciences, 325, 227-236.

[22] Klement, E.P., & Mesiar, R., & Pap, E. (2000). Triangular Norms, Kluwer Academic Publishers, Dordrecht.

[23] Kerre, E. E., & Nachtegael, M. (2000). Fuzzy techniques in image processing, in Studies in Fuzziness and Soft Computing, Heidelberg, Germany: Physica-Verlag, 52.

[24] Klir, G. J., & Yuan, B. (1995). Fuzzy Sets and Fuzzy Logic, Theory and Applications, Englewood Cliffs, NJ: Prentice Hall.

[25] Mas, M., & Mayor, G., & Torrens, J. (1999). t-operators, Int.J.Uncertain. Fuzz.Knowl. -BasedSyst. 7, 31–50.

[26] Mayor, G., & Torrens,J. (1993). On a class of operators for expert systems, Int. J. Intell. Syst., 8 771–778.

[27] Sun, X., & Liu, H. (2020). Representation of nullnorms on bounded lattices, Information Sciences, 539, 269-276.

[28] Xie, J., & Ji, W.(2021). New Constructions of Nullnorms on Bounded Lattices, Journal of Applied Mathematics and Physics, 9, 1-10.

15. A NOTE ON THE SRIVASTAVA SINGHAL POLYNOMIALS

Yahya ÇİN **Nejla ÖZMEN**

Abstract

This study is related to Srivastava-Singhal plinomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$. Some generating functions for Srivastava-Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ are obtained. Some properties are given. Graphs are drawn for special values.

Keywords. Srivastava Singhal polynomials, Generating function, Bilinear and bilateral generating functions, Recurrence relations

ORCID: 0009-0009-9571-6071
e-mail: yahya23117@ogr.duzce.edu.tr

ORCID: 0000-0001-7555-1964
e-mail: nejlaozmen@duzce.edu.tr

1 Introduction

The Srivastava Singhal polynomials $\mathfrak{G}_d^{(\alpha)}(\nu, p, \omega, s)$ are defined by generating relation (see, [13], Eq. (3.2), p. 78)

$$\sum_{d=0}^{\infty} \mathfrak{G}_d^{(\alpha)}(\nu, p, \omega, s)\, t^d = (1-st)^{-\frac{\alpha}{s}} \exp\left\{ w\nu^p \left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \tag{1.1}$$

where $\alpha >$ -1, s is a positive integer.

It is from (1.1) that [13],

$$\mathfrak{G}_d^{(\alpha)}(\nu,p,\omega,s) = \frac{s^d}{d!}\sum_{j=0}^{\infty}\frac{e^{w\nu^p}(-w)^j}{j!}\left(\frac{\alpha+pj}{s}\right)_d \nu^{pj} \quad (1.2)$$
$$= \frac{s^d}{d!}\sum_{z=0}^{\infty}\frac{(-w\nu^p)^z}{z!}$$
$$x\sum_{j=0}^{z}(-1)^{z-j}\binom{z}{j}\left(\frac{\alpha+pj}{s}\right)_d$$

where $(\omega)_\varpi$ denotes the Pochhammer symbol defined by

$$(\omega)_\varpi = \begin{cases} \frac{\Gamma(\omega+\varpi)}{\Gamma(\omega)} = \omega(\omega+1)\dots(\omega+\varpi-1), & \varpi = 1,2,3,\dots \\ (\omega)_0 = 1. & \end{cases}$$

These polynomials are given by the following generating relation (see, [1], pp.431):

$$\sum_{d=0}^{\infty}\binom{d+h}{d}\mathfrak{G}_{d+h}^{(\alpha)}(\nu,p,\omega,s)\,t^d \quad (1.3)$$
$$= (1-st)^{\frac{-(\alpha+hs)}{s}}\exp\left\{w\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\}$$
$$x\mathfrak{G}_h^{(\alpha)}\left(\nu(1-st)^{-\frac{1}{s}},p,\omega,s\right).$$

Now our recall of the relationship ([14], p. 315, Eq. (83)):

$$Y_d^{\alpha}(u;s) = s^{-d}\mathfrak{G}_d^{(\alpha+1)}(u,1,1,s) \ \ (\alpha > -1; \ \ s = 1,2,...)$$

where $Y_d^{\alpha}(u;s)$ denotes the Konhauser biorthogonal polynomials (cf. [15, 16, 17, 18, 19]). In particular,

$$Y_d^{\alpha}(\nu;1) = L_d^{(\alpha)}(\nu) = \mathfrak{G}_n^{(\alpha+1)}(\nu,1,1,s) \quad (1.4)$$
$$(\alpha > -1; \ \ s = 1,2,...)$$

and The polynomials $Y_d^{\alpha}(\nu;2)$ were previously encountered by Spencer and Fano [20] in certain analytical calculations involving gamma-ray penetration of matter.

In Chapter 2, the sum expression of Srivastava Singhal polynomial $\mathfrak{G}_n^{(\alpha)}(\nu,p,\omega,s)$ and some values and graphs corresponding to these values will be given.

In Chapter 3 we prove several theorems concerning different families of generating functions for the polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ using the method studied by Chen and Srivastava [12].
In addition, in Chapter 4, as an application of these theorems, we present some generating relations for the Srivastava-Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ which are given by (1.2). Some miscellaneous recurrence relations of the Srivastava-Singhal polynomials given by (1.1) are given in the last section.

2 Some Properties of Srivastava Singhal Polynomials

In this section, the sum expression will be given with the help of Srivastava Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ theorem. Expressions corresponding to some values of Srivastava Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ will be given and the graphs of these expressions will be given.

Theorem 2.1. We have

$$\begin{aligned} & \mathfrak{G}_d^{(\alpha_1+\alpha_2)}(\nu, p, \omega_1 + w_2, s) \\ = & \sum_{h=0}^{d} \mathfrak{G}_{d-h}^{(\alpha_1)}(\nu, p, w_1, s)\, \mathfrak{G}_h^{(\alpha_2)}(\nu, p, w_2, s)\,. \end{aligned} \tag{2.2}$$

Proof. Replacing α by $\alpha_1 + \alpha_2$ and w by $w_1 + w_2$ in (1.1) we obtain

$$\begin{aligned} & \sum_{d=0}^{\infty} \mathfrak{G}_d^{(\alpha_1+\alpha_2)}(\nu, p, \omega_1 + w_2, s)\, t^d \\ = & (1-st)^{-\frac{(\alpha_1+\alpha_2)}{s}} \exp\left\{(\omega_1 + w_2)\,\nu^p \left[\left(1-(1-st)^{-\frac{p}{s}}\right)\right]\right\} \\ = & (1-st)^{-\frac{\alpha_1}{s}} \exp\left\{w_1 \nu^p \left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\ & \times (1-st)^{-\frac{\alpha_2}{s}} \exp\left\{w_2 \nu^p \left[\left(1-(1-st)^{-\frac{p}{s}}\right)\right]\right\} \\ = & \sum_{d=0}^{\infty} \mathfrak{G}_d^{(\alpha_1)}(\nu, p, \omega_1, s)\, t^d \sum_{h=0}^{\infty} \mathfrak{G}_h^{(\alpha_2)}(\nu, p, \omega_2, s)\, t^h \\ = & \sum_{d=0}^{\infty} \sum_{h=0}^{d} \mathfrak{G}_{d-h}^{(\alpha_1)}(\nu, p, \omega_1, s)\, \mathfrak{G}_h^{(\alpha_2)}(\nu, p, \omega_2, s)\, t^d. \end{aligned}$$

Matching the coefficients of the first and last members yields the desired identity (2.2). □

A few of Srivastava Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ defined (1.1) and (1.2) are:

$$\begin{aligned}
\mathfrak{G}_0^{(\alpha)}(\nu,1,1,1) &= 1, \\
\mathfrak{G}_1^{(\alpha)}(\nu,1,1,1) &= -\nu+\alpha, \\
\mathfrak{G}_2^{(\alpha)}(\nu,1,1,1) &= \frac{1}{2}\left[\nu^2-\nu(2\alpha+2)+\alpha^2+\alpha\right], \\
\mathfrak{G}_3^{(\alpha)}(\nu,1,1,1) &= \frac{1}{6}\left[-\nu^3+3\nu^2(\alpha+2)-3\nu\left(\alpha^2+3\alpha+2\right)+\alpha^3+3\alpha^2+2\alpha\right].
\end{aligned}$$

The following shows the graphs of these polynomials (up to $(n,s) = (1,1)$) in special case $\alpha = 0,1,2,3,4$ are shown below:

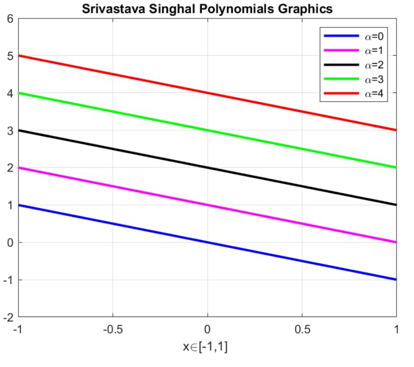


Figure 30:

The following shows the graphs of these polynomials (up to $(n, s) = (2, 1)$) in special case $\alpha = 0, 1, 2, 3, 4$ are shown below:

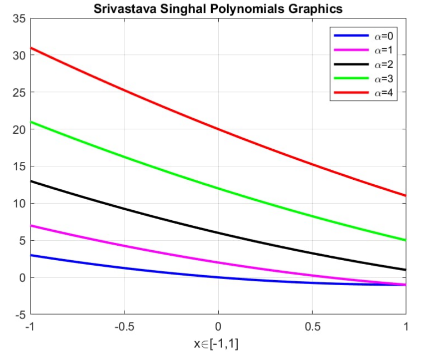


Figure 31:

The following shows the graphs of these polynomials (up to $(n, s) = (3, 1)$) in special case $\alpha = 0, 1, 2, 3, 4$ are shown below:

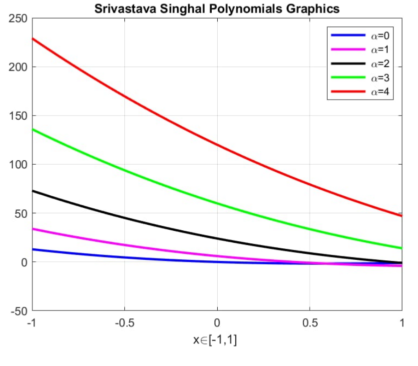


Figure 32:

3 Generating Functions

This section discuss several substantial families of bilinear and bilateral generating functions for the multivariable multiparameter Srivastava-Singhal polynomials $\mathfrak{G}_n^{(\alpha)}(\nu, p, \omega, s)$ in (1.2). By using the similar method considered in (see, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11])

Our starting point is the theorem below.

Theorem 3.1. Let

$$\Pi_{\mu,\varrho}\left[\mathfrak{y}_1,...,\mathfrak{y}_r;\xi\right]:=\sum_{d=0}^{\infty}a_d\varphi_{\mu+\varrho d}(\mathfrak{y}_1,...,\mathfrak{y}_r)\xi^d,\ (a_d\neq 0).$$

Suppose also that

$$\begin{aligned}&\Theta_{h,p}^{\mu,\varrho}\left(\nu,p,\omega,s;\mathfrak{y}_1,...,\mathfrak{y}_r;\xi\right)\\ :\quad &=\sum_{d=0}^{[h/p]}a_d\mathfrak{G}_{h-pd}^{(\alpha)}\left(\nu,p,\omega,s\right)\varphi_{\mu+\varrho d}(\mathfrak{y}_1,...,\mathfrak{y}_r)\xi^d.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{h=0}^{\infty}\Theta_{h,p}^{\mu,\varrho}\left(\nu,p,\omega,s;\mathfrak{y}_1,...,\mathfrak{y}_r;\frac{\eta}{t^p}\right)t^h\\ =\quad&(1-st)^{-\frac{\alpha}{s}}\exp\left\{w\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\}\Pi_{\mu,\varrho}\left[\mathfrak{y}_1,...,\mathfrak{y}_r;\eta\right].\end{aligned}\tag{3.2}$$

Proof. Let S denotes the left side (3.2). Then we find

$$S=\sum_{h=0}^{\infty}\sum_{d=0}^{[h/p]}a_d\mathfrak{G}_{h-pd}^{(\alpha)}\left(\nu,p,\omega,s\right)\varphi_{\mu+\varrho d}(\mathfrak{y}_1,...,\mathfrak{y}_r)\eta^dt^{h-pd}.\tag{3.3}$$

Now, setting $h\to h+pd$ in (3.3), we obtain

$$S=\sum_{h=0}^{\infty}\sum_{d=0}^{\infty}a_d\mathfrak{G}_{h}^{(\alpha)}\left(\nu,p,\omega,s\right)\varphi_{\mu+\varrho d}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)\eta^dt^h.$$

Then by the generating relation (1.1), we find

$$S=(1-st)^{-\frac{\alpha}{s}}\exp\left\{w\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\}\Pi_{\mu,\varrho}\left[\mathfrak{y}_1,\ldots,\mathfrak{y}_r;\eta\right].$$

□

Theorem 3.4. Let

$$\begin{aligned}&\Lambda_{n,q,\mu,\psi}^{\alpha_1,\alpha_2}\left(\nu,p,\omega_1+\omega_2,s;\mathfrak{y}_1,\ldots,\mathfrak{y}_r;t\right)\\ :\quad&=\sum_{d=0}^{[n/q]}a_d\mathfrak{G}_{n-qd}^{(\alpha_1+\alpha_2)}\left(\nu,p,\omega_1+\omega_2,s\right)\varphi_{\mu+\varrho d}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)t^d\end{aligned}$$

where $a_d \neq 0$, $n, q \in \mathbb{N}$ and the notation $[n/q]$ means the largest integer that is less than or equal to n/q.

Then, for $q \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{d=0}^{n} \sum_{l=0}^{[d/q]} a_l \mathfrak{G}_{n-d}^{(\alpha_1)} (\nu, p, \omega_1, s)\, \mathfrak{G}_{d-ql}^{(\alpha_2)} (\nu, p, \omega_2, s)\, \varphi_{\mu+\varrho l} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^l \\ = \; &\Lambda_{n,q,\mu,\varrho}^{\alpha_1,\alpha_2} (\nu, p, \omega_1 + \omega_2, s; \mathfrak{y}_1, \ldots, \mathfrak{y}_r; z) \end{aligned} \tag{3.5}$$

provided that every member of (3.5) exists.

Proof. Let Δ denote the first member of the assertion (3.5) of Theorem 3.4. Then, upon subsituting for the polynomials $\mathfrak{G}^{(\alpha_1+\alpha_2)} (\nu, p, \omega_1 + \omega_2, s)$ from the (2.2) into the left-hand side of (3.5), we obtain

$$\begin{aligned} \Delta &= \sum_{d=0}^{n} \sum_{l=0}^{[d/q]} a_l \mathfrak{G}_{n-d}^{(\alpha_1)} (\nu, p, \omega_1, s)\, \mathfrak{G}_{n-ql}^{(\alpha_2)} (\nu, p, \omega_2, s)\, \varphi_{\mu+\varrho l} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^l \\ &= \sum_{l=0}^{[n/q]} \sum_{d=0}^{n-ql} a_l \mathfrak{G}_{n-d-ql}^{(\alpha_1)} (\nu, p, \omega_1, s)\, \mathfrak{G}_{d}^{(\alpha_2)} (\nu, p, \omega_2, s)\, \varphi_{\mu+\varrho l} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^l \\ &= \sum_{l=0}^{[n/q]} a_l \left(\sum_{d=0}^{n-ql} \mathfrak{G}_{n-d-ql}^{(\alpha_1)} (\nu, p, \omega_1, s)\, \mathfrak{G}_{d}^{(\alpha_2)} (\nu, p, \omega_2, s) \right) \varphi_{\mu+\varrho l} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^l \\ &= \sum_{l=0}^{[n/q]} a_l \mathfrak{G}_{n-ql}^{(\alpha_1+\alpha_2)} (\nu, p, \omega_1 + \omega_2, s)\, \varphi_{\mu+\varrho l} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^l \\ &= \Lambda_{n,q,\mu,\psi}^{\alpha_1,\alpha_2} (\nu, p, \omega_1 + \omega_2, s; \mathfrak{y}_1, \ldots, \mathfrak{y}_r; z) . \end{aligned}$$

□

Theorem 3.6. Let

$$\begin{aligned} &\varepsilon_{m,h,\mu,d} [\nu, p, \omega, s; \mathfrak{y}_1, \ldots, \mathfrak{y}_r; z] \\ : \; &= \sum_{u=0}^{\infty} a_u \mathfrak{G}_{m+hu}^{(\alpha)} (\nu, p, \omega, s) \varphi_{\mu+du} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^u, \end{aligned}$$

where $a_u \neq 0$ and

$$\Xi_{m,h,\mu} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r; z) := \sum_{\delta=0}^{[u/h]} \binom{m+u}{u-h\delta} a_\delta \varphi_{\mu+d\delta} (\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^\delta.$$

Then, for $h, \mu \in \mathbb{N}$; we have

$$
\begin{aligned}
&\sum_{u=0}^{\infty} \mathfrak{G}_{m+u}^{(\alpha)}(\nu, p, \omega, s) \Xi_{m,d,h}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r; z) t^u \\
= \quad & (1-st)^{-\frac{(\alpha+ms)}{s}} \exp\left\{\omega \nu^p \left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\
& \times \varepsilon_{m,h,\mu,d}\left[\nu (1-st)^{-\frac{1}{s}}, p, \omega, s; \mathfrak{y}_1, \ldots, \mathfrak{y}_r; z\left(\frac{t}{1-st}\right)^h\right] \quad (3.7)
\end{aligned}
$$

Proof. Let κ denote the first member of the assertion (3.7) of proposition 3.6. Then,

$$
\kappa = \sum_{u=0}^{\infty} \mathfrak{G}_{u+m}^{(\alpha)}(\nu, p, \omega, s) \sum_{\delta=0}^{[u/h]} \binom{m+u}{u-h\delta} a_j \varphi_{\mu+d\delta}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^\delta t^u.
$$

Replacing u by $u + h\delta$ and then using $(3.7\ )$, we may write that

$$
\begin{aligned}
\kappa \quad = \quad & \sum_{u=0}^{\infty}\sum_{\delta=0}^{\infty} \binom{m+u+h\delta}{u} \mathfrak{G}_{u+m+h\delta}^{(\alpha)}(\nu, p, \omega, s)\, a_\delta \varphi_{\mu+d\delta}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r) z^\delta t^{u+h\delta} \\
= \quad & \sum_{\delta=0}^{\infty}\left(\sum_{u=0}^{\infty} \binom{m+u+h\delta}{u} \mathfrak{G}_{u+m+h\delta}^{(\alpha)}(\nu, p, \omega, s)\, t^u\right) a_\delta \varphi_{\mu+d\delta}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r)\left(zt^h\right)^\delta \\
= \quad & \sum_{\delta=0}^{\infty} a_\delta (1-st)^{-\frac{(a+h\delta s+mu)}{s}} \exp\left\{\omega\nu^c\left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\
& \times \mathfrak{G}_{m+h\delta}^{(\alpha)}\left(\nu(1-st)^{-\frac{1}{s}}, p, \omega, s\right) \varphi_{\mu+d\delta}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r)\left(zt^h\right)^\delta \\
= \quad & (1-st)^{-\frac{(a+ms)}{s}} \exp\left\{\omega\nu^c\left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\
& \times \sum_{\delta=0}^{\infty} a_\delta \mathfrak{G}_{m+h\delta}^{(\alpha)}\left(\nu(1-st)^{-\frac{1}{s}}, p, \omega, s\right) \varphi_{\mu+d\delta}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r)\left(\frac{zt^h}{(1-st)^h}\right)^\delta \\
= \quad & (1-st)^{-\frac{(a+ms)}{s}} \exp\left\{\omega\nu^c\left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\
& \times \varepsilon_{m,h,\mu,d}\left[\nu(1-st)^{-\frac{1}{s}}, p, \omega, s; \mathfrak{y}_1, \ldots, \mathfrak{y}_r; z\left(\frac{t}{1-st}\right)^h\right],
\end{aligned}
$$

the end of the proof. □

4 Special Cases

Further applications of the above theorems can be made by expressing the multivariable functions $\varphi_{\mu+\varrho d}\left(\mathfrak{y}_1, \ldots, \mathfrak{y}_r\right)$, $d \in \mathbb{N}_0$, $r \in \mathbb{N}$, in terms of simpler functions of one and more variables. We first set

$$
\varphi_{\mu+\varrho d}\left(\mathfrak{y}_1, \ldots, \mathfrak{y}_r\right) = \Theta_{\mu+\varrho d}^{(\alpha)}\left(\mathfrak{y}_1, \ldots, \mathfrak{y}_r\right)
$$

in Theorem 3.1, where the multivarible polynomials extension of the multivariable polynomials $\varphi_{\mu+\varrho d}^{(\alpha)}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)$ [3], generated by

$$\left(1-\nu_1\mathfrak{z}\right)^{-\alpha}e^{(\nu_2+\ldots+\nu_r)\mathfrak{z}}=\sum_{n=0}^{\infty}\Theta_n^{(\alpha)}\left(\nu_1,\ldots,\nu_r\right)\mathfrak{z}^n \tag{4.1}$$

$$\left(|\mathfrak{z}|<|\nu_1|^{-1}\right).$$

This leads to the following result. It gives a class of bilateral generating functions for the multivariable extension of the multivariable polynomials $\Theta_{\mu+\varrho d}^{(\alpha)}\left(\nu_1,\ldots,\nu_r\right)$ and the Srivastava-Singhal polynomials.

Corollary 4.2. If

$$\Pi_{\mu,\varrho}\left[\mathfrak{y}_1,\ldots,\mathfrak{y}_r;\xi\right]:=\sum_{d=0}^{\infty}a_d\Theta_{\mu+\varrho d}^{(\alpha)}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)\xi^d$$
$$\left(a_d\neq 0,\ \mu,\varrho\in\mathbb{C}\right),$$

then, we have

$$\begin{aligned}&\sum_{h=0}^{\infty}\sum_{d=0}^{[h/p]}a_d\mathfrak{G}_{h-pd}^{(\alpha)}\left(\nu,p,\omega,s\right)\Theta_{\mu+\varrho d}^{(\alpha)}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)\frac{\xi^d}{t^{pd}}t^h\\ =\;&(1-st)^{-\frac{\alpha}{s}}\exp\left\{w\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\}\Pi_{\mu,\varrho}\left[\mathfrak{y}_1,\ldots,\mathfrak{y}_r;\xi\right].\end{aligned}$$

Remark 4.3. Using the generating relation (4.1) for the multivariable polynomials $\Theta_{\mu+\varrho d}^{(\alpha)}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)$ and getting $a_d=1$, $\mu=0$, $\varrho=1$ in Corollary 4.2, we find that

$$\begin{aligned}&\sum_{h=0}^{\infty}\sum_{d=0}^{[h/p]}\mathfrak{G}_{h-pd}^{(\alpha)}\left(\nu,p,\omega,s\right)\Theta_{d}^{(\alpha)}\left(\mathfrak{y}_1,\ldots,\mathfrak{y}_r\right)\xi^dt^{h-pd}\\ =\;&(1-st)^{-\frac{\alpha}{s}}\exp\left\{w\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\}\left\{\left(1-\mathfrak{y}_1\xi\right)^{-\alpha}e^{(\mathfrak{y}_2+\ldots+\mathfrak{y}_r)\xi}\right\},\end{aligned}$$

$$\left(|\xi|<|\mathfrak{y}_1|^{-1}\right).$$

If we set $r=1$, $\mathfrak{y}_1=\mathfrak{y}$ and

$$\varphi_{\mu+\varrho d}\left(\mathfrak{y}\right)=\mathfrak{G}_{\mu+\varrho d}^{(\alpha_3)}\left(\mathfrak{y},\mathfrak{c},\mathfrak{q}_3,\mathfrak{w}\right)$$

in Theorem 3.4, we have the following bilinear generating functions for the Srivastava Singhal polynomials.

Corollary 4.4. If

$$\begin{aligned}
&\Lambda_{n,q,\mu,\varrho}^{\alpha_1,\alpha_2}\left(\nu,p,\omega_1+\omega_{2,}s;\mathfrak{y},\mathfrak{c},\mathfrak{q}_{3,}\mathfrak{w};t\right)\\
&:\ =\sum_{d=0}^{[n/q]} a_d \mathfrak{G}_{n-qd}^{(\alpha_1+\alpha_2)}\left(\nu,p,\omega_1+\omega_{2,}s\right)\mathfrak{G}_{\mu+\varrho d}^{(\alpha_3)}\left(\mathfrak{y},\mathfrak{c},\mathfrak{q}_3,\mathfrak{w}\right)t^d
\end{aligned}$$

where $a_d \neq 0$, μ, $\varrho \in \mathbb{C}$. Then, we get

$$\begin{aligned}
&\sum_{d=0}^{n}\sum_{l=0}^{[d/q]} a_l \mathfrak{G}_{n-d}^{(\alpha_1)}\left(\nu,p,\omega_1,s\right)\mathfrak{G}_{d-ql}^{(\alpha_2)}\left(\nu,p,\omega_2,s\right)\mathfrak{G}_{\mu+\varrho l}^{(\alpha_3)}\left(\mathfrak{y},\mathfrak{c},\mathfrak{q}_3,\mathfrak{w}; z\right)\\
&= \Lambda_{n,q,\mu,\varrho}^{\alpha_1,\alpha_2}\left(\nu,p,\omega_1+\omega_2,s;\mathfrak{y},\mathfrak{c},\mathfrak{q}_3,\mathfrak{w};z\right)
\end{aligned} \tag{4.5}$$

provided that there are any number of (4.5).

Remark 4.6. If we take $a_l = 1$, $\mu = 0$, $\varrho = 1$, $q = 1$, $z = 1$, $\mathfrak{y} = \nu$, $\mathfrak{c} = p$, $\mathfrak{q}_3 = \omega_3$, $\mathfrak{w} = s$ and then use the relation (2.2) for Srivastava Singhal polynomials in Corollary 4.4, we have

$$\begin{aligned}
&\sum_{d=0}^{n}\sum_{l=0}^{d}\mathfrak{G}_{n-d}^{(\alpha_1)}\left(\nu,p,\omega_1,s\right)\mathfrak{G}_{d-l}^{(\alpha_2)}\left(\nu,p,\omega_2,s\right)\mathfrak{G}_{l}^{(\alpha_3)}\left(\nu,p,\omega_3,s\right)\\
&= \sum_{d=0}^{n}\mathfrak{G}_{n-d}^{(\alpha_1)}\left(\nu,p,\omega_1,s\right)\sum_{l=0}^{d}\mathfrak{G}_{d-l}^{(\alpha_2)}\left(\nu,p,\omega_2,s\right)\mathfrak{G}_{l}^{(\alpha_3)}\left(\nu,p,\omega_3,s\right)\\
&= \sum_{d=0}^{n}\mathfrak{G}_{n-d}^{(\alpha_1)}\left(\nu,p,\omega_1,s\right)\mathfrak{G}_{d}^{(\alpha_2+\alpha_3)}\left(\nu,p,\omega_2+\omega_3,s\right)\\
&= \mathfrak{G}_{n}^{(\alpha_1+\alpha_2+\alpha_3)}\left(\nu,p,\omega_1+\omega_2+\omega_3,s\right).
\end{aligned}$$

So we get a family of bilateral generating functions for the generalized Cesàro polynomials and the Srivastava-Singhal polynomials as follows:

Corollary 4.7. If

$$\begin{aligned}
&\varepsilon_{m,h,\mu,d}\left[\nu,p,\omega,s;\lambda,\mathfrak{y};z\right] := \sum_{u=0}^{\infty} a_u \mathfrak{G}_{m+hu}^{(\alpha)}(\nu,p,\omega,s) g_{\mu+du}^{(\sigma)}\left(\lambda,\mathfrak{y}\right)z^u\\
&\left(a_u \neq 0,\ m \in \mathbb{N}_0,\ \mu,d \in \mathbb{C}\right)
\end{aligned}$$

and

$$\Xi_{m,d,h}\left(\lambda,\mathfrak{y};z\right) := \sum_{\delta=0}^{[u/h]}\binom{m+u}{u-h\delta} a_\delta g_\delta^{(\sigma)}\left(\lambda,\mathfrak{y}\right)z^\delta$$

where $d, h \in \mathbb{N}$, then we have

$$\begin{aligned}
&\sum_{u=0}^{\infty} \mathfrak{G}_{m+u}^{(\alpha)}(\nu, p, \omega, s)\Xi_{m,d,h}(\lambda, \mathfrak{y}; z)t^u \\
=\ & (1-st)^{-\frac{(\alpha+ms)}{s}} \exp\left\{\omega\nu^p\left[1-(1-st)^{-\frac{p}{s}}\right]\right\} \\
&\times\varepsilon_{m,h,\mu,d}\left[\nu(1-st)^{-\frac{1}{s}}, p, \omega, s; \lambda, \mathfrak{y}; z\left(\frac{t}{1-st}\right)^h\right] \quad (4.8)
\end{aligned}$$

on the condition that each member (4.8) exists.

Note that the statements of Theorem 3.1, Theorem 3.4, Theorem 3.6 can be applied to obtain many different families of multilinear and multilateral generating functions for the Srivastava Singhal polynomials, for any appropriate choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable functions $\varphi_{\mu+\psi k}(\mathfrak{y}_1, \ldots, \mathfrak{y}_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler relative functions.

5 Recurrence Relations

We will now discuss some miscellaneous recurrence relations of the Srivastava-Singhal polynomials given by (1.1). If we differentiate each member of the generating function relation (1.1) with respect to ν and using

$$\sum_{d=0}^{\infty}\sum_{h=0}^{\infty} P(h,d) = \sum_{d=0}^{\infty}\sum_{h=0}^{d} P(h, d-h),$$

gives the following (differential) recurrence relation for the Srivastava Singhal polynomials given explicitly by (1.1):

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\mathfrak{G}_d^{(\alpha)}(\nu, p, \omega, s) = \omega p\nu x^{p-1}\mathfrak{G}_d^{(\alpha)}(\nu, p, \omega, s) \quad (5.1) \\
&-\omega p\nu^{p-1}\sum_{m=0}^{d}\left(\frac{p}{s}\right)_m \frac{s^m}{m!}\mathfrak{G}_{d-m}^{(\alpha)}(\nu, p, \omega, s).
\end{aligned}$$

Using the relation (1.4) and getting $\alpha \to \alpha+1, p=\omega=s=1$ in (5.1). The (differential) recurrence relation for Laguerre polynomials is

$$\frac{d}{d\nu}L_d^{(\alpha)}(\nu) = L_d^{(\alpha)}(\nu) - \sum_{m=0}^{d} L_{d-m}^{(\alpha)}(\nu).$$

Similarly, the special case $\alpha \to \alpha+1, p=\omega=1$ of (5.1). For the biorthogonal Konhauser polynomials we have the following (differential) recurrence relation

$$\frac{d}{d\nu}Y_d^{(\alpha)}(\nu;s)=Y_d^{(\alpha)}(\nu;s)-\sum_{m=0}^{d}\left(\frac{1}{s}\right)_m\frac{1}{m!}Y_{d-m}^{(\alpha)}(\nu;s).$$

Another recurrence relation for these polynomials is obtained by differentiating each number of the generating functions of the relation (1.1) with respect to t,:

$$(d+1)\,\mathfrak{G}_{d+1}^{(\alpha)}(\nu,p,\omega,s)=\alpha\sum_{m=0}^{d}s^m\mathfrak{G}_{d-m}^{(\alpha)}(\nu,p,\omega,s) \quad (5.2)$$

$$-\omega p\nu^p\sum_{v=0}^{d}\left(\frac{p}{s}+1\right)_v\frac{s^v}{v!}\mathfrak{G}_{d-m}^{(\alpha)}(\nu,p,\omega,s).$$

If we choose $\alpha \to \alpha+1, p=\omega=s=1$ in (5.2), we get the following recurrence relation for the Laguerre polynomials:

$$(d+1)\,L_{d+1}^{(\alpha+1)}(\nu)=(\alpha+1)\sum_{m=0}^{d}L_{d-m}^{(\alpha+1)}(\nu)-\nu\sum_{v=0}^{d}(v+1)\,L_{d-v}^{(\alpha+1)}(\nu).$$

Writing v instead of m; we may write that

$$(d+1)\,L_{d+1}^{(\alpha+1)}(\nu)=\sum_{v=0}^{d}(\alpha+1-\nu v-\nu)\,L_{d-v}^{(\alpha+1)}(\nu).$$

Finally, if we make $\alpha \to \alpha+1, p=\omega=1$ in (5.2), we obtain establish the following recurrence relation for the biorthogonal Konhauser polynomials:

$$\begin{aligned}(d+1)\,Y_{d+1}^{\alpha+1}(\nu;s) &= (\alpha+1)\sum_{m=0}^{d}s^mY_{d-m}^{\alpha+1}(\nu;s)\\ &\quad -\nu\sum_{p=0}^{d}\left(\frac{1}{s}+1\right)_p\frac{s^p}{p!}Y_{d-p}^{\alpha+1}(\nu;s).\end{aligned}$$

References

[1] Srivastava H. M. & Manocha H. L.(1984). A Treatise on Generating Functions. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York.

[2] Altın A. & Erkuş E. (2006). On a multivariable extension of the Lagrange-Hermite polynomials. Integral Transforms and Special Functions. 17(4), 239-244.

[3] Özmen N. & Erkus-Duman E. (2013). Some results for a family of multivariable polynomials. AIP Conf. Proc. 1558, 1124-1127.

[4] Özmen N. & Erkus-Duman E. (2015). On the Poisson-Charlier polynomials. Serdica Math. J. 41, 457-470.

[5] Aktaş R. & Erkus-Duman E. (2013). On a family of multivariate modifed Humbert polynomials. The Scientic World Journal, page 12.

[6] Özmen N. & Erkus-Duman E. (2018). Some families of generating functions for the generalized Cesáro polynomials. J. Comput. Anal. Appl. 25(4), 670-683.

[7] Erkuş E. & Srivastava H. M. (2006). A unified presentation of some families of multivariable polynomials. Integral Transforms Spec. Funct. 17, 267-273.

[8] Srivastava H. M., Özarslan M. A. & Kaanoğlu C. (2013). Some Generalized Lagrange-Based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Russian Journal of Mathematical Physics. 20(1), 110-120.

[9] Korkmaz-Duzgun D. & Erkuş-Duman E. (2018). The Laguerre type d-orthogonal polynomials. Journal of Science and Arts. 1(42) , 95-106.

[10] Özmen N. & Çin Y. (2017). On The Konhauser biorthogonal polynomials. Turk. J. Math. Comput. Sci. 7, 48-55.

[11] Aktaş R. & Erkuş-Duman E. (2013). The Laguerre polynomials in several variables. Mathematica Slovaca. 63(3), 531-544.

[12] Chen K.-Y. & Srivastava H. M. (2005). Series identities and associated families of generating functions. J. Math. Anal. Appl. 311, 582-599.

[13] Srivastava H. M. & Singhal J. P. (1971). A class of polynomials defined by generalized Rodrigues' formula, Ann. Mat. Pura Appl., 90(1), 75-85.

[14] Srivastava H. M. & Lavoie J.-L. (1975). A certain method of obtaining bilateral generating functions. Indagationes Mathematicae (Proceedings). 78(4), 304-320.

[15] Konhauser J. D. E. (1965) Some properties of biorthogonal polynomials. J. Math. Anal. Appl. 11, 242-260.

[16] Konhauser J. D. E. (1967). Biorthogonal polynomials suggested by the Laguerre polynomials. Pasific J. Math. 21(2), 303-314.

[17] Samanta K. P. & Samanta B. (2015). On bilateral generating functions of Konhauser Biorthogonal polynomials. Universal Journal of Applied Math. 3(2), 18-23.

[18] Preiser S. (1962). An investigation of biorthogonal polynomials derivable from ordinary differential equations of the third order. J. Math. Anal. Appl. 4, 38-64.

[19] Srivastava H. M. (1982). Some biorthogonal polynomials suggested by the Laguerre polynomials. Pacific J. Math. 98(1), 235-250.

[20] Spencer L. & Fano U. (1951). Penetration and diffusion of X-rays. Calculation of spatial distribution by polynomial expansion, J. Res. Nat. Bur. Standards, 46(6), 446-461.

16. ON BALANCING NUMBERS AND APPLICATION TO CODING THEORY

Vedat Kabasakal **Fatih Yılmaz**

Abstract

This paper aims to give a new approach coding theory via the well-known balancing numbers. Additionally, it describes how error-correcting codes are designed and implemented, and how these codes contribute to the robustness of the system.

Keywords. Balancing Array, Lucas-Balancing Sequence, Coding Theory Error Correction code

ORCID: 0000-0001-7873-1979
e-mail: fatih.yilmaz@hbv.edu.tr
ORCID: 0000-0001-6449-1070
e-mail: vedatkabasakal@gmail.com

1 Introduction

Coding theory and cryptography are often thought to mean the same thing. Coding theory is concerned with encoding information into different symbols. Cryptography, on the other hand, uses codes to create messages that only certain people can read. Cryptographers work on strategies to make it difficult for those without additional information to "break" the codes. Coding theory, however, ignores the question of who can access the code and how secret it can be. The primary goal of the theory is to detect and correct errors in your code [12, 3].

Coding theory is focused on the transmission of data over noisy channels and the restoration of corrupted messages in the process. The concept of a noisy channel can be defined as small alterations or distortions that may occur in the message [4].

In 1996, NASA sent the Pathfinder robot to explore the surface of Mars. This robot attempted to send reliable, raw information to a world hundreds of millions of kilometers away using radio waves powerful enough to light a simple bulb. This transmission of information was made possible through coding theory, which combines computer science and mathematical structure [1]

Claude Shannon's work titled *A Mathematical Theory of Communication* published in 1948, established a solid foundation for the theory and popularized it [11]. This study mathematically demonstrated that a noisy communication channel has a numerical value called channel capacity, and with appropriate coding and decoding techniques, reliable communication can be achieved at or below the channel capacity. However, while Shannon's proof suggested the existence of suitable coding, it did not provide a clear method for how to achieve this. Subsequently, coding theory evolved with research on how to perform good coding. The first steps were taken by Richard W. Hamming, who published the details of his research on error-correcting codes. Coding theory then grew rapidly over a little more than half a century, attracting the interest of not only electronic engineers and computer scientists but also mathematicians [1].

In recent years, a new concept in integer sequences, known as Balancing numbers, first appeared when Behera and Panda [2] were investigating integer solutions of a first-degree Diophantine equation. They discovered the Balancing numbers "n" and the cobalancing numbers "r".

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

For example, Balancing numbers such as 6, 35, 204, 1189, 6930, and their corresponding cobalancing numbers 2, 14, 84, 492, 2870 have been identified.

Definition 1.1. The Balancing sequence, with initial conditions $B_0 = 0, B_1 = 1$, satisfies the following recurrence relation
[1, 5]:

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1$$

The sequence $\{B_n\} = \{0, 1, 6, 35, 204, 1189, 6930, 40391, 235416 \ldots\}$ is cataloged as OEIS A001109[10]. The closed-form or Binet's formula for the sequence is defined as:

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

The auxiliary matrix that gives the balancing numbers is a 2x2 matrix [9].

$$B = \begin{pmatrix} 6 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} B_2 & B_1 \\ -B_1 & B_0 \end{pmatrix}.$$

Since $\det(B) = 1$, it is suitable for use in coding theory applications:

$$B^n = \begin{pmatrix} B_{n+1} & B_n \\ -B_n & B_{n-1} \end{pmatrix}.$$

Definition 1.2. The Lucasbalancing sequence C_n is defined with initial conditions $C_1 = 3, C_2 = 17$ and the recurrence relation
[9, 6].

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2.$$

The Lucas-balancing numbers share the same recurrence relation as the nalancing numbers but with different initial conditions. The Binet's formula for the Lucas Balancing numbers is given by:

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}.$$

The balancing and Lucas balancing sequences can also be defined for negative indices as follows:

$$B_{-n} = 6B_{-n+1} - B_{-n+2} = -B_n,$$
$$C_{-n} = 6C_{-n+1} - C_{-n+2} = C_n.$$

Furthermore, the recurrence relation between the Balancing and Lucas-balancing sequences is given by:

$$C_{n+1} = 3C_n - 8B_n, \quad n \geq 0.$$

Behera and Panda defined the following limits:

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3 + \sqrt{8},$$

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 3 + \sqrt{8},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

In Ray's (2015) paper, the matrix $S = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$ is defined. For the Balancing sequence $\{B_n\}$ and the Lucas Balancing sequence $\{C_n\}$, for all integers n, Ray gave the following relation:

$$S^k = \begin{pmatrix} C_k & 8B_k \\ B_k & C_k \end{pmatrix}.$$

Theorem 1.3. For the Balancing sequence $\{B_n\}$ and the Lucas Balancing sequence $\{C_n\}$, we have:

$$S^{-k} = \begin{pmatrix} C_k & -8B_k \\ -B_k & C_k \end{pmatrix}.$$

Proof. Since $S^k = \begin{pmatrix} C_k & 8B_k \\ B_k & C_k \end{pmatrix}$ and $\det(S) = 1$, we obtain:

$$S^{-k} = \frac{1}{\det(S)} \begin{pmatrix} C_k & -8B_k \\ -B_k & C_k \end{pmatrix} = \begin{pmatrix} C_k & -8B_k \\ -B_k & C_k \end{pmatrix}.$$

□

Theorem 1.4. For the Balancing sequence $\{B_n\}$ and the Lucas Balancing sequence $\{C_n\}$, we have:

$$\lim_{n\to\infty} \frac{C_n}{B_n} = \sqrt{8}.$$

Proof. Dividing both sides of the equation $C_{n+1} = 3C_n - 8B_n$ by C_n:

$$\frac{C_{n+1}}{C_n} = 3 - \frac{8B_n}{C_n}.$$

Taking the limit as n approaches infinity on both sides of the equation:

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 3 + 8 \lim_{n\to\infty} \frac{B_n}{C_n}.$$

Using the relation from equation (3):

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 3 + \sqrt{8}.$$

Therefore, we obtain:

$$\lim_{n\to\infty} \frac{C_n}{B_n} = \sqrt{8}.$$

□

2 Coding Algorithm

Let us place our message into a matrix M by replacing the space between two words with the symbol "!". We then divide the message matrix into 2×2 block matrices M_i. For an arbitrary number k, we create the matrix S^k. The message matrix for an arbitrary number n is obtained using the Table 1.

A	**B**	**C**	**D**	**E**
n	n+1	n+2	n+3	n+4
F	**G**	**H**	**I**	**J**
n+5	n+6	n+7	n+8	n+9
K	**L**	**M**	**N**	**O**
n+10	n+11	n+12	n+13	n+14
P	**Q**	**R**	**S**	**T**
n+15	n+16	n+17	n+18	n+19
U	**V**	**W**	**X**	**Y**
n+20	n+21	n+22	n+23	n+24
Z	**,**	**!**	**0**	**?**
n+25	n+26	n+27	n+28	n+29

Table 13: *Alphabet Table*

The encoded message matrix is found as $E = M \times S^n$. Since $\det S = 1$, the operation can easily be reversed.

$$\det E = \det(M \times S^k) = \det M \times \det S^k = \det M \times 1 = \det M$$

Given:

$$M_i = \begin{bmatrix} m_1^i & m_2^i \\ m_3^i & m_4^i \end{bmatrix}$$

$$S^k = \begin{bmatrix} C_k & 8B_k \\ B_k & C_k \end{bmatrix}$$

$$E_i = \begin{bmatrix} e_1^i & e_2^i \\ e_3^i & e_4^i \end{bmatrix}$$

$$E = M \times S^k = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \times \begin{bmatrix} C_k & 8B_k \\ B_k & C_k \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$$

From this, the following equalities are obtained. It can be easily seen that $e_1, e_2, e_3, e_4 \geq 0$ since the elements of the message matrix and the S matrix are positive.

$$e_1 = m_1 \cdot C_k + m_2 \cdot B_k \geq 0 \quad (1)$$
$$e_2 = m_1 \cdot 8B_k + m_2 \cdot C_k \geq 0 \quad (2)$$
$$e_3 = m_3 \cdot C_k + m_4 \cdot B_k \geq 0 \quad (3)$$
$$e_4 = m_3 \cdot 8B_k + m_4 \cdot C_k \geq 0 \quad (4)$$

If (2) is divided:

$$\frac{C_k}{8B_k} \leq \frac{e_1}{e_2} \leq \frac{B_k}{C_k}$$

From Theorem 1:

$$\frac{e_2}{e_1} \approx \sqrt{8}$$

Similarly:

$$\frac{e_4}{e_3} \approx \sqrt{8}$$

$$\frac{e_1}{e_3} = \frac{e_2}{e_4} = \frac{m_1 + m_2}{m_3 + m_4}.$$

Decoding Algorithm

The message matrix M is obtained by multiplying the encoded matrix E by the matrix S^{-k}:

$$M = E \times S^{-k} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \times \begin{bmatrix} C_k & -8B_k \\ -B_k & C_k \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$$

If $\det M = \det E$, it is understood that the message was received without errors. Otherwise, error correction is applied.

Error Correction Let's assume there is a single error:

$$E' = \begin{bmatrix} a & e_2 \\ e_3 & e_4 \end{bmatrix}$$

Here, a is the corrupted element of the message. The corrupted element can be corrected using:

$$a \cdot e_4 - e_3 \cdot e_2 = \det C$$

Similarly, if b, c, or d are corrupted elements, the error is corrected as follows:

$$E' = \begin{bmatrix} e_1 & b \\ e_3 & e_4 \end{bmatrix}, E' = \begin{bmatrix} e_1 & e_2 \\ c & e_4 \end{bmatrix}, E' = \begin{bmatrix} e_1 & e_2 \\ e_3 & d \end{bmatrix}$$

Now, let's consider the case of a double error:

$$E' = \begin{bmatrix} a & b \\ e_3 & e_4 \end{bmatrix}$$

Let a and b be the corrupted elements of the message. Since $b/a \approx \sqrt{8}$, the double error can also be corrected. Now, consider the case of a triple error:

$$E' = \begin{bmatrix} a & b \\ c & e_4 \end{bmatrix}$$

Using similar reasoning, $b/a \approx \sqrt{8}$ and $c/e_4 \approx \sqrt{8}$, and with the equation $a \cdot e_4 - b \cdot c = \det C$, solutions for triple errors can be found. There are 16 possible cases ($2^4 = 16$). If all elements are correct, no error correction is needed. If all elements of the message are found to be incorrect, it means that 93.33% ($14/15 = 0.9333$) of the errors have been corrected.

3 Illustrative Application

Let our message be "MERHABA NASILSIN" Here, we will replace the space with the "!" symbol.

$$M = \begin{bmatrix} M & E & R & H \\ A & B & A & ! \\ N & A & S & I \\ L & S & I & N \end{bmatrix}_{4\times 4}$$

We can divide the 4x4 matrix M into block matrices M_i (where $1 \leq i \leq 4$) from left to right, each with a size of 2x2:

$$M_1 = \begin{bmatrix} M & E \\ A & B \end{bmatrix}, \quad M_2 = \begin{bmatrix} R & H \\ A & ! \end{bmatrix}, \quad M_3 = \begin{bmatrix} N & A \\ L & S \end{bmatrix}, \quad M_4 = \begin{bmatrix} S & I \\ I & N \end{bmatrix}$$

Let our arbitrary value of n be 5. In this case, we can find the corresponding numbers for our message using the alphabet table.

M	E	R	H	A	B	A	!
17	9	22	12	5	6	5	32
N	A	S	I	L	S	I	N
18	5	23	13	16	23	13	18

Table 14: *Sample Message Matrix Values*

$$M_1 = \begin{bmatrix} 17 & 9 \\ 5 & 6 \end{bmatrix}, M_2 = \begin{bmatrix} 22 & 12 \\ 5 & 32 \end{bmatrix}, M_3 = \begin{bmatrix} 18 & 5 \\ 16 & 23 \end{bmatrix}, M_4 = \begin{bmatrix} 23 & 13 \\ 13 & 18 \end{bmatrix}$$

For the arbitrary value $k = 6$, the following is found:

$$S^6 = \begin{bmatrix} C_6 & 8B_6 \\ B_6 & C_6 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}^6 = \begin{bmatrix} 19601 & 55440 \\ 6930 & 19601 \end{bmatrix}$$

$$E_1 = M \times S^6 = \begin{bmatrix} 17 & 9 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 19601 & 55440 \\ 6930 & 19601 \end{bmatrix} = \begin{bmatrix} 395587 & 1118889 \\ 139585 & 394806 \end{bmatrix}$$

The determinant $\det E = 57$ is sent to the recipient along with the matrix by adding a column at the beginning of the matrix. The matrix sent is:

$$E_1^* = \begin{bmatrix} 5 & 395587 & 1118889 \\ 6 & 139585 & 394806 \\ 57 & 0 & 0 \end{bmatrix}$$

Similarly, the matrices M_2, M_3, and M_4 are also sent to the recipient. A stronger result is obtained if different values of n and k are chosen for each process. Tables must be referenced in the text with their table number.

Decoding

The matrix E_1^* is given as:

$$E_1^* = \begin{bmatrix} 5 & 395587 & 1118889 \\ 6 & 139585 & 394806 \\ 57 & 0 & 0 \end{bmatrix}$$

From the first column of the matrix E_1^*, the values $n = 5$ and $k = 6$ are found.

The value of S^{-6} is found as follows:

$$S^{-6} = \begin{bmatrix} C_6 & -8B_6 \\ -B_6 & C_6 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -1 & 3 \end{bmatrix}^6 = \begin{bmatrix} 19601 & -55440 \\ -6930 & 19601 \end{bmatrix}$$

$$M = E \times S^{-6} = \begin{bmatrix} 395587 & 1118889 \\ 139585 & 394806 \end{bmatrix} \times \begin{bmatrix} 19601 & -55440 \\ -6930 & 19601 \end{bmatrix} = \begin{bmatrix} 17 & 9 \\ 5 & 6 \end{bmatrix}$$

Since det $E =$ det M, it is understood that the message was received without errors. When $n = 5$, using the alphabet table: 17 = M, 9 = E, 5 = A, and 6 = B are found. Thus:

$$M_1 = \begin{bmatrix} 17 & 9 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} M & E \\ A & B \end{bmatrix}$$

Similarly, the M matrix is constructed using the block matrices.

4 RESULTS

A coding method using a matrix S that involves the relationship between balancing numbers and Lucas-balancing numbers forms the basis of the algorithm. This coding method is based on matrix multiplication, a well-known algebraic operation widely applied in modern computers. The fundamental practical feature of this method lies in its ability to detect errors in large units of information, especially within the matrix elements. The elements of the initial matrix M, and therefore the elements of the code matrix E, can be numbers with unlimited values. This theoretically means that an unlimited number of values can be adjusted with this coding method. The correction power of this method is approximately 93%.

References

[1] Arda, D., (2011). Kodlama Teorisinin Kriptografik Açıdan Incelenmesi, Doktara Tezi, Trakya Üniversitesi .Edirne,

[2] Behera, A., & Panda, G. K. (1999). On the square roots of triangular numbers. Fibonacci Quarterly, 37, 98-105.

[3] Eraşık, M. E. (2014). Balans Sayıları ve Cebirsel Özellikleri Doktora Tezi, Bursa Uludag Üniversitesi.

[4] Kalkan, M., & Zararsız, Z. (2021). Kriptografi, steganografi ve kodlama teorisinin grup teorisi ile ilişkisinin incelenmesi ve bazı uygulamaların geliştirilmesi. Nevşehir: Nevşehir Hacı Bektaş Veli Üniveritesi Fen Bilimleri Enstitüsü.

[5] Liptai, K., Panda, G. K., & Szalay, L. (2016). A balancing problem on a binary recurrence and its associate. The Fibonacci Quarterly, 54, 235-241.

[6] Prasad, B. (2018). Coding theory on balancing numbers. Int. J. Open Problems Compt. Math, 11(4), 73-85.

[7] Prasad, B. (2021). Coding theory based on balancing polynomials. Control and Cybernetics, 50(2), 335-346.

[8] Ray, P. K. (2009). Balancing and cobalancing numbers (Doctoral dissertation).

[9] Ray, P. K. (2015). Balancing and Lucas-balancing sums by matrix methods. Mathematical Reports, 17(2), 225-233.

[10] Sloane, N. J. (2003). The on-line encyclopedia of integer sequences.OEIS

[11] Shannon, C.E. (1948) A Mathematical Theory of Communication. Bell System Technical Journal, 27, 379–423.

[12] Özkan, Mustafa (2009). Kodlama Teorisi Üzerine. Yayımlanmamış Yüksek Lisans Tezi, Edirne: Trakya Üniversitesi.

17. A REVIEW OF MATHEMATICAL MODELS OF WOUND HEALING PROCESS

Rıdvan Yaprak

Abstract

A wound is a disruption of skin integrity due to trauma or disease. Wound healing involves the formation of new healthy tissue to replace damaged tissue. It generally occurs in four stages: hemostasis, inflammation, proliferation, and remodelling. These stages may vary according to the location of the wound. In this study, we intend to provide a brief overview of the mathematical models of the wound healing process. In this context, we present the mathematical models that describe the four stages of wound healing. Additionally, this review presents ordinary differential equation models that compare normal and diabetic healing in full-thickness wounds and investigate the relationship between slough (dead tissue) and wound in chronic healing conditions.

Keywords. Mathematical models of wound healing process, Ordinary differential equations

ORCID: 0000-0002-6417-4929
e-mail: ridvanyaprak@ktu.edu.tr

1 Introduction

The disruption of the integrity of the skin is termed a wound [1]. In daily life, many individuals experience wounds due to physical impacts on the body, which typically heal within a reasonable timeframe. However, the healing rate of some wounds can be very slow, and in some scenarios, no healing occurs. Wound healing is the process through which damaged tissue is replaced with new healthy tissue [1, 2] The wound healing process is intricate and dynamic, progressing through defined stages. Generally, it unfolds in four stages, and these stages can vary based on the wound site and circumstances [3]. The stages of this healing process are known as hemostasis, inflammation, proliferation, and remodeling, sequentially.

The initial phase of the wound healing process, known as the hemostasis phase, starts right after the injury, during which the body's first physiological responses to halt bleeding occur. During the second phase, the inflammatory phase (lasting 2-3 days for skin wounds), inflammatory cells are dispatched to the site of the injury. Neutrophils are the first responders to the wound site, where they remove damaged and dead tissue. Subsequently, monocyte cells arrive at the wound and initiate macrophage formation. Macrophages play a pivotal role in wound healing, aiding in phagocytosis and antimicrobial defense during the inflammation phase. The inflammation phase concludes with the entry of the last lymphocyte cells to the wound. The third phase, the proliferation phase (occurring 3-10 days for skin wounds), marks the onset of granulation tissue formation, a key indicator of wound healing. At this juncture, fibroblast cells migrate from adjacent healthy tissues towards the wound site. Meanwhile, endothelial cells draw blood from healthy venules near the wound and develop new capillaries. Epithelial cells move towards each other from different directions, facilitating granulation formation. The fourth phase, the remodeling stage (spanning from 21 days to 1 year for skin wounds), involves collagen build-up in the wound and a resultant decrease in wound tension and surface area [3]. The stages of the healing process are discussed in detail in [2, 4, 5].

Wounds are classified into two categories, acute and chronic, based on their healing duration [6]. Acute wounds typically heal quickly and with fewer complications, whereas chronic wounds have an interrupted healing process, adversely affecting the individual's quality of life [1, 6]. Moreover, treating chronic wounds imposes a substantial financial burden on healthcare systems [7, 8]. In chronic wounds, where healing is prolonged, a barrier-forming tissue called slough develops, hindering wound recovery [9]. Particularly in the case of chronic wounds, this necrotic tissue must be periodically removed and the wound area cleansed. This removal process, known as debridement [10], is also performed for acute wounds, typically just once to clean the wound site [11]. Serial debridement, which can be employed using various techniques, significantly enhances the wound healing process [12, 13, 14, 15].

This brief review introduces a mathematical model for each stage of healing [16, 17]. Furthermore, it covers an ordinary differential equation model describing the interaction between the wound and the slough [18]. Lastly, an ordinary differential equation model that investigates the healing processes in full-thickness wounds for both normal and diabetic cases is presented [19].

2 Mathematical Models of Healing Stages

This section presents a mathematical model for each stage of the healing process.

2.1 Homeostasis Phase: Mathematical Modelling of Blood Coagulation

Blood coagulation is a crucial phase within hemostasis where solid clots are created. This process is vital for hemostasis as it involves covering a damaged vessel wall with a clot composed of platelets and fibrin, thus halting bleeding and initiating the repair of the damaged vessel. In this section, the model introduced in [16] is presented. The model is expressed as a system of ordinary differential equations as follows.

$$
\begin{aligned}
\frac{IXa}{dt} &= k_1\beta - h_1 IXa \\
\frac{VIIIa}{dt} &= k_2 IIa + k_3 Xa - k_4 APC\frac{VIIIa}{b_1 + VIIIa} - h_2 VIIIa \\
\frac{Xa}{dt} &= k_5 IXa\frac{VIIIa}{b_2 + VIIIa} - h_3 Xa \\
\frac{Va}{dt} &= k_6 IIa - k_7 APC\frac{Va}{b_3 + Va} - h_4 Va \\
\frac{APC}{dt} &= k_8 IIa - h_5 APC \\
\frac{IIa}{dt} &= k_9 Xa\frac{Va}{b_4 + Va} - h_6 IIa
\end{aligned}
\tag{2.1}
$$

where

a : means activated versions
β : the concentration of the coagulant that activates factor IX
IXa : the concentration of plasma thromboplastin
$VIIIa$: the concentration of antihemophilic factor
Xa : the concentration of Stuart-Power factor
Va : the concentration of Labile factor
APC : the concentration of protein C
IIa : the concentration of prothrombin
k_i : kinetic constant of i-th reaction
h_i : inactivation constant of i-th factor.

The nonlinear system of ordinary differential equations (2.1) represents the mathematical model of blood coagulation. The primary objective of this model is to determine the relative speeds of the reactions necessary for coagulation. The results of numerical simulations performed using the "ode15s" solver in MATLAB are as follows.

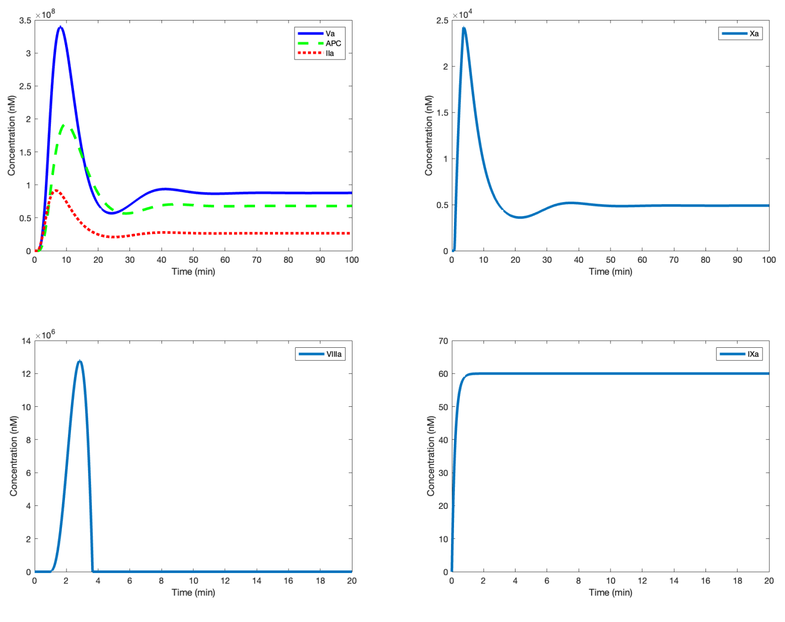


Figure 33: *Numerical solution of (2.1) model*

The numerical outcomes shown in Figure 33 indicate that factors $VIIIa$ and IXa are excessively rapid within the reaction scheme. When a wound occurs, factor IXa quickly initiates the coagulation process.

Parameter	Value
k_1	15
h_1	6.12
k_2	2.3
k_3	1.4
k_4	4
b_1	11
h_2	0.35
k_5	221
b_2	23
h_3	0.33
k_6	2.6
k_7	0.5
b_3	6.1
h_4	0.4
k_8	0.87
h_5	0.34
k_9	1900
b_4	250
h_6	0.35

Table 15: *Paramater values of (2.1) model for numerical simulation*

2.2 Inflammation Phase: Mathematical Model of Inflammatory Cells

In the inflammation phase, inflammatory cells migrate to the wound site. Examples of these cells include Mac-1 and F4/80.

Hypothesize that there is an attractive signal that will attract and concentrate inflammatory cells at the wound site. Moreover, the cell count diminishes since they are unable to undergo further division in that area and thus cannot aid in cell proliferation. Based on these premises, the mathematical model suggested in [17] is as follows.

$$\frac{dN}{dt} = k_1 e^{-at} - k_2 N. \tag{2.2}$$

The analytical solution of (2.2) is as follows.

$$N(t) = \frac{k_1}{k_2 - a}(e^{-at} - e^{-k_2 t}) + N(0). \tag{2.3}$$

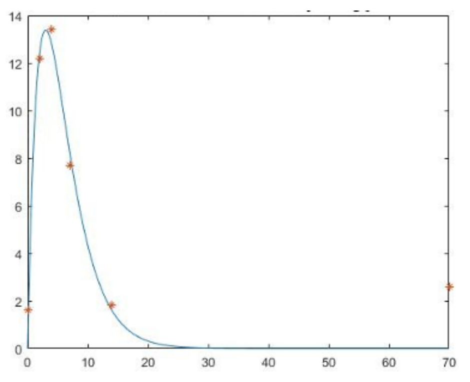


Figure 34: *Numerical solution of (2.2)*

As seen in Figure 34, the number of inflammatory cells initially increases rapidly while in the healing phase, then decreases and approaches zero.

2.3 Proliferation Phase: Mathematical Modeling of Collagen Accumulation

At the beginning of the proliferation phase, collagen accumulation occurs due to the release of amino acids due to the fibroblast effect and tissue degradation.

The assumptions of the collagen accumulation model proposed in [17] are as follows.

1. At $t = 0$, the concentration of collagen is zero.

2. The maximum concentration of collagen is A.

3. Initially, the rate of collagen accumulation is maximum.

4. As time evolves, the rate of collagen accumulation decreases and approaches zero towards the end of the healing process.

The collagen accumulation model is expressed as an ordinary differential equation as follows.

$$\begin{aligned} \frac{dc}{dt} &= k(A-c) \\ c(0) &= 0, c(\infty) = A \end{aligned} \tag{2.4}$$

The analytical solution of (2.4) is as follows.

$$c(t) = A(1 - e^{-kt}). \tag{2.5}$$

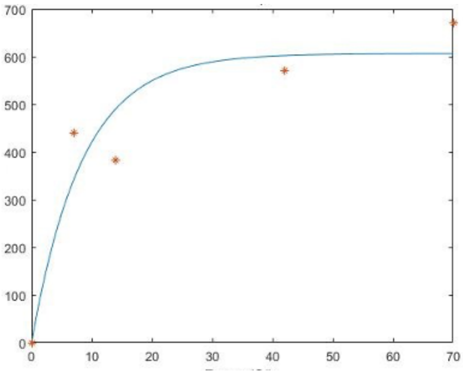


Figure 35: *Numerical solution of (2.4)*

As seen in Figure 35, the numerical solution of the model satisfies all assumptions.

2.4 Remodelling Phase: Mathematical Modelling of Tensile Strenght of The Wound

As healing progresses, the wound space is filled with new tissue components, leading to an increase in tensile strength. The assumptions of the tensile strength model proposed in [17]) are as follows.

1. At $t = 0$, there is no tensile strength.
2. As healing progresses, the tensile strength increases.
3. The maximum tensile strength is A.

The tensile strength model is expressed as an ordinary differential equation as follows.

$$\frac{dS}{dt} = kS\left(1 - \frac{S}{A}\right)$$
$$S(0) = 0 \tag{2.6}$$

The analytical solution of (2.6) is as follows.

$$S(t) = \frac{A}{1 + Be^{-kt}}, B = \frac{A}{S(0)}. \tag{2.7}$$

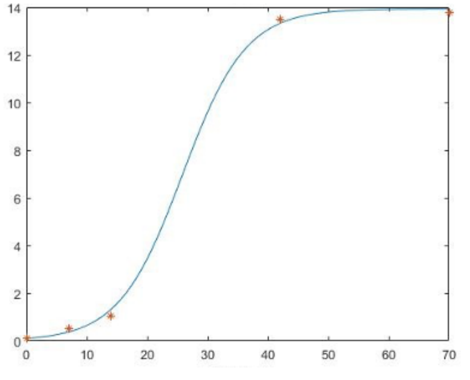


Figure 36: *Numerical solution of (2.6)*

As seen in Figure 36, the numerical solution of the model satisfies all assumptions.

3 Wound-Slough Interaction Model

In chronic wounds with a slow healing process, a structure called dead tissue (slough) is formed, which acts as a barrier to wound healing (Angel, 2019). The first mathematical model that analyzes the interaction between wounds and slough was introduced in [18].

The physiological assumptions of the model are as follows.

1. As the wound area increases, the slough area also increases.

2. If there is slough, the area of the wound decreases quickly.

3. The interaction of the slough and the wound causes the wound area to increase.

4. The slough decreases on its own.

The wound and slough interaction is described below by a set of nonlinear differential equations.

$$\begin{aligned} \frac{dA}{dt} &= -hA + pAS \\ \frac{dS}{dt} &= rA - \delta S \\ A(0) &= A_0, S(0) = S_0. \end{aligned} \tag{3.1}$$

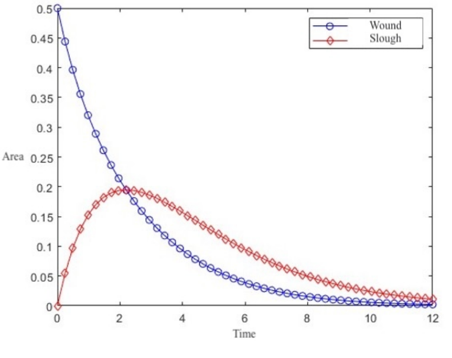


Figure 37: *Numerical solution of (3.1) with* $A(0) = 0.5, S(0) = 0, \delta, h, p, r = 0.5$

As seen in Figure 37, the wound heals over time. This numerical simulation is an example of acute wound healing.

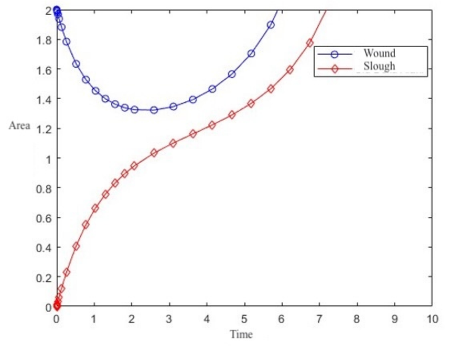


Figure 38: *Numerical solution of (3.1) with* $A(0) = 2, S(0) = 0, \delta, h, p, r = 0.5$

The numerical solutions in this section are obtained using the MATLAB ode solver "ode45". As illustrated in Figure 38, wounds deteriorate over time. This numerical simulation exemplifies a chronic wound.

4 A Mathematical Model for Wound Healing and the Impact of Diabetes

In this section we present an ODE model for wound healing, pertaining to the dominant processes that contribute to the closure of a full-thickness wound [19]. The model is represented below as a nonlinear system of ordinary differential equations.

$$\begin{aligned}\frac{dA_e}{dt} &= \lambda(1+\nu-A_e)\left(\frac{A_e-\gamma A_d}{1-\gamma A_d}\right)\\ \frac{dA_s}{dt} &= -\left(\beta_0+\beta_1\left(\frac{A_s-A_d}{1+\nu-A_d}\right)\right)(1+\nu-A_s)A_s\\ \epsilon\frac{A_d}{dt} &= A_S-A_d-\alpha H(t-t_c)A_d.\end{aligned} \quad (4.1)$$

Here, A_e represents the wound area in the epidermis, A_s represents the wound area in the subdermis, and A_d represents the wound area in the dermis.

The numerical solutions obtained using the MATLAB ode solver "ode15s" of (4.1) with the values of the diabetic and non-diabetic parameters in Table 16 are given below.

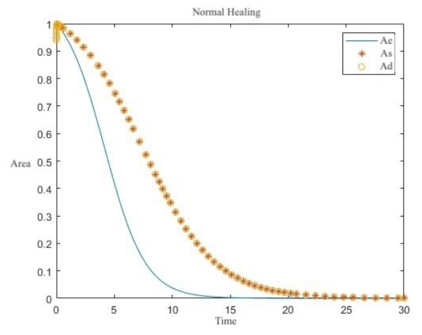


Figure 39: *Numerical solution of (4.1 (Normal Healing)*

As seen in Figure 39, the wound heals completely with the values of the non-diabetic parameters.

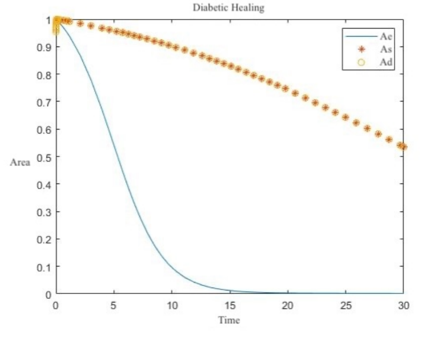


Figure 40: *Numerical solution of (4.1 (Diabetic Healing)*

As seen in Figure 40, the area of the wound constantly increases with the values of diabetic parameters and the wound does not heal.

Parameter	Physical Interoretation	Non-diabetic	Diabetic
λ	Epidermal growth rate	0.5274	0.4258
β_0	Basal dermal growth rate	0.2950	0.0715
β_1	Mechanosensitive dermal growth rate	0.0124	0.0026
α	Contraction with respect to elastic response	0.2144	0.1593
γ	Epidermal dependence on dermis	0.0027	0.1868
ν	Proportion of proliferative region at wound margin	0.0904	0.1
ϵ	Tethering with resoect to elastic response	0.01	0.01
t_c	Contraction switch delay	3	3
θ	Contraction switch gradient	1	1
K_d	Ratio of initial to recoiled wound area	0.0904	0.9542

Table 16: *Parameter meanings and values of (4.1)*

5 Conclusion and Acknowledgment

Numerous models in the literature explore various healing processes from multiple perspectives. This brief review presents several mathematical models that represent wound healing processes. Moving forward, our goal is to refine the models presented in this review and to develop new models that examine different aspects of the healing process, along with performing analyses on these improved models.

The initial advancement performed in this context pertains to the interaction model between wounds and sloughs, which is also covered in this review and was included as part of the author's doctoral dissertation. Therefore, the author expresses gratitude to his Ph.D. supervisor Prof. Dr. Erhan Coşkun.

References

[1] Baktır, G. (2019). Wound repair and experimental wound models. Experimed, 9(3), 130-137.

[2] Nguyen, D., T., Orgill, D., P., & Murphy, G.,F. (2009).The pathophysiologic basis for wound healing and cutaneous regeneration: Biomaterials for Treating Skin Loss, Springer.

[3] Werner, S. & Grose, R. (2003). Regulation of wound healing by growth factors and cytokines. Physiological Reviews 83, 835-870.

[4] Velnar, T., Bailey, T. & Smorkolj, V. (2009). The wound healing process: an overview of the cellular and molecular mechanisms. The Journal of International Medical Research 37, 1528–1542.

[5] Kirsner, R., S. & Eaglstein, W., H. (1993). The wound healing process. Dermatologic Clinics 11, 629–640.

[6] Orgill, D., P. & Blanco. (2009). Biomaterials for treating skin loss. Woodhead Publishing Ltd..

[7] Nussbaum, S., R., Carter, M., J., Fife, C., E., DaVanzo, J., Haught, R., Nusgart, M. & Cartwright, D. (2018). An economic evaluation of the impact, cost, and medicare policy implications of chronic nonhealing wounds. Value in Health 21, 27–32.

[8] Fife, C., E., Carter, M., J., Walker, D. & Thomson, B. (2012). Wound care outcomes and associated cost among patients treated in us outpatient wound centers: Data from the us wound registry. Wounds 24, 10–17.

[9] Angel, D. (2019). Slough: what does it mean and how can it be managed. Wound Practice and Research 27.

[10] Bradley, M., Cullum, N., & Sheldon, T. (2001). The debridement of chronic wounds: a systematic review. Health Technology Assessment 3.

[11] Yanar, F. & Baktıroğlu, S. (2015). Surgical treatment in acute infections. Journal of TOTBİD 14, 392–397.

[12] Cardinal, M., Eisenbud, D., E., Armstrong, D., G., Zelen, C., Driver, V., Attinger, C., Phillips, T., & Harding, K. (2009). Serial surgical debridement: A retrospective study on clinical outcomes in chronic lower extremity wounds. Wound Repair and Regeneration 17, 306–311.

[13] Doerler,M.,Reich-Schupke,S.,Altmeyer,P. & Stücker,M. (2012). Impact on wound healing and efficacy of various leg ulcer debridement techniques. JDDG: Journal der Deutschen Dermatologischen Gesellschaft 10, 624–631.

[14] Liu, W.,L., Jiang, Y.,L., Wang, Y.,Q., Li, Y.,X. & Liu, Y.,X. (2017). Combined debridement in chronic wounds: A literature review. Chinese Nursing Research 4, 5–8.

[15] Thomas, D., C., Tsu, C., L., Nain, R., A., Arsat, N., Fun, S., S. & Lah, N. A. S. N. (2021). The role of debridement in wound bed preparation in chronic wound: A narrative review. Annals of Medicine and Surgery 71, 2049–0801.

[16] Sahin, D., Riedel, I., Muzhinji, K., Daukšte, L., & Zhelezova, M. (2007). Team 6 Mathematical Modeling of Blood Coagulation.

[17] Bardsley, W., Sattar, A., Armstrong, J., Shah, M., Brosnan, P., & Ferguson, M. (1995). Quantitative analysis of wound healing. Wound Repair and Regeneration 3, 426-441.

[18] Jones, M.,A., Song, B. & Thomas, D.,M. (2004). Controlling Wound Healing Through Debridement, Math. Comput. Model. 40, 1057-1064.

[19] Bowden, L.,G., Maini, P., K., Moulton, D.E., Tang, J.,B., Liu, P., Y. & Byrne, H., M. (2014). An ordinarty differential equation model for full thickness wounds and the effects of diabetes. Journal of Theoritical Biology 36, 87-100.

18. ON ENCRYPTION/DECRYPTION WITH PERRIN NUMBERS

Satı Yılmaz **Fatih Yılmaz**

Abstract

This paper focuses on encryption/decryption by exploiting properties of the Perrin numbers. This method exhibits an impressive 99.80% error correction capability.

Keywords. Perrin numbers, Recurrence relations, Coding theory, Error-correcting codes, Perrin sequence, Recursive sequences in coding, Error detection

ORCID: 0009-0008-2911-3742
e-mail: sati.yilmaz@ogr.hbv.edu.tr
ORCID: 0000-0001-7873-1979
e-mail: fatih.yilmaz@hbv.edu.tr

1 Introduction

In coding theory, the investigation of special number sequences and their applications has led to remarkable discoveries and practical applications. The Perrin sequence comprises a series of integers defined by a simple recurrence relation and it is the sequence A001608, in [6] . For $n \geq 3$, the sequence of Perrin numbers $\{P_n\}_{n\geq 0}$ is defined by the recurrence relation,

$$P_{n+3} = P_{n+1} + P_n, \tag{1.1}$$

where $P_0 = 3, P_1 = 0, P_2 = 2$ are initial values. The negative indexed Perrin numbers [1] are defined by the following recurrence relation:

$$P_{-n} = P_{-n+3} - P_{-n+1}.$$

In Table 1, please find some values of the Perrin numbers.

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
P_n	-1	-2	4	-3	2	1	-1	3	0	2	3	2	5	5	7

Table 17: *Perrin numbers*

In 1899, Raoul Perrin introduced the sequence, focusing on its combinatorial properties. The characteristic equation of Perrin numbers is given as below,

$$x^3 - x - 1 = 0$$

for details, please visit [3] and [4]. In [7], the Padovan Q-matrix is defined as below:

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and it is proved that

$$Q^n = \begin{bmatrix} \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_n \\ \mathcal{P}_n & \mathcal{P}_{n+2} & \mathcal{P}_{n+1} \\ \mathcal{P}_{n+1} & \mathcal{P}_{n+3} & \mathcal{P}_{n+2} \end{bmatrix}$$

where $\mathcal{P}_n$ is the nth Padovan number. The authors [8] developed matrix arrays representing Padovan and Perrin numbers and made comparisons between Padovan and Perrin matrix sequences. In [2], the authors obtained the Padovan and Perrin numbers that are concatenations of two terms of the other sequence. The Perrin sequence is not extensively studied in the area of coding theory. In [5], J. Shtayat et. al. give a model of cryptography based on the Padovan Q-matrix and Perrin R-matrix using the blocking method.

At this paper, inspired by recent advancements, we select a message and encode and decode it using Perrin numbers. Lastly, we investigate the mechanisms for error detection and correction when an incorrect message is delivered to the receiver.

2 Perrin Numbers and its application to Coding Theory

The properties of the Perrin sequence, including its recursive nature and distinct numerical properties, make it a valuable tool in designing error-correcting codes with desirable properties.

Theorem 2.1. For $n \geq 1$,

$$KQ^n = D_n \tag{2.2}$$

where

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, K = \begin{bmatrix} P_{-5} & P_{-3} & P_{-4} \\ P_{-4} & P_{-2} & P_{-3} \\ P_{-3} & P_{-1} & P_{-2} \end{bmatrix},$$

$$D_n = \begin{bmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}. \tag{2.3}$$

Proof. Let us prove it by using Principle of Mathematical Induction. For $n = 1$, it is easy to see that the statement verifies. Assume the theorem holds for the case $n = k$, i.e.:

$$KQ^k = D_k.$$

Now, we have to prove the theorem for $n = k + 1$. That is:

$$KQ^{k+1} = D_{k+1}.$$

It can be rewritten as follows:

$$KQ^{k+1} = KQ^k Q.$$

Using the induction hypothesis and matrix multiplication, it is easy to verify the result. So, the proof is completed.

□

Example 2.4. For $n = 3$,

$$KQ^3 = \begin{bmatrix} P_{-5} & P_{-3} & P_{-4} \\ P_{-4} & P_{-2} & P_{-3} \\ P_{-3} & P_{-1} & P_{-2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^3$$
$$= \begin{bmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 3 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} P_{-2} & P_0 & P_{-1} \\ P_{-1} & P_1 & P_0 \\ P_0 & P_2 & P_1 \end{bmatrix} = D_3.$$

For $n = 4$,

$$\begin{aligned} KQ^4 &= \begin{bmatrix} P_{-5} & P_{-3} & P_{-4} \\ P_{-4} & P_{-2} & P_{-3} \\ P_{-3} & P_{-1} & P_{-2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^4 \\ &= \begin{bmatrix} 4 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} P_{-1} & P_1 & P_0 \\ P_0 & P_2 & P_1 \\ P_1 & P_3 & P_2 \end{bmatrix} = D_4. \end{aligned}$$

Theorem 2.5. For $n \geq 1$;

$$\det D_n = 23.$$

Proof.

$$\begin{aligned} \det D_n &= \det(KQ^n) = \det(K)\det(Q)^n \\ &= 23. \end{aligned}$$

□

Theorem 2.6. For $n \geq 1$;

$$(D_n)^{-1} = \frac{1}{|D_n|} \begin{bmatrix} P_{n-2}^2 - P_{n-3}P_{n-1} & P_{n-4}P_{n-1} - P_{n-3}P_{n-2} & P_{n-3}^2 - P_{n-4}P_{n-2} \\ P_{n-3}^2 - P_{n-4}P_{n-2} & P_{n-5}P_{n-2} - P_{n-4}P_{n-3} & P_{n-4}^2 - P_{n-5}P_{n-3} \\ P_{n-4}P_{n-1} - P_{n-2}P_{n-3} & P_{n-3}^2 - P_{n-5}P_{n-1} & P_{n-5}P_{n-2} - P_{n-3}P_{n-4} \end{bmatrix}.$$

Proof. By exploiting the well-known formula, the inverse of $(D_n)^{-1}$ can be obtained, easily:

$$(D_n)^{-1} = \frac{1}{det(D_n)} adj(D_n).$$

□

Example 2.7. For $n = 3$:

$$\begin{aligned} (D_3)^{-1} &= \begin{bmatrix} -\frac{6}{23} & -\frac{2}{23} & \frac{9}{23} \\ \frac{9}{23} & \frac{3}{23} & -\frac{2}{23} \\ -\frac{2}{23} & \frac{7}{23} & \frac{3}{23} \end{bmatrix} \\ &= \frac{1}{\det D_3} \begin{bmatrix} P_1^2 - P_0P_2 & P_{-1}P_2 - P_0P_1 & P_0^2 - P_{-1}P_1 \\ P_0^2 - P_{-1}P_1 & P_{-2}P_1 - P_{-1}P_0 & P_{-1}^2 - P_{-2}P_0 \\ P_{-1}P_2 - P_1P_0 & P_0^2 - P_{-2}P_2 & P_{-2}P_1 - P_0P_{-1} \end{bmatrix} \\ &= \frac{1}{23} \begin{bmatrix} -6 & -2 & 9 \\ 9 & 3 & -2 \\ -2 & 7 & 3 \end{bmatrix}. \end{aligned}$$

Example 2.8. For $n = 4$:

$$
\begin{aligned}
(D_4)^{-1} &= \begin{bmatrix} \frac{4}{23} & \frac{9}{23} & -\frac{6}{23} \\ -\frac{6}{23} & -\frac{2}{23} & \frac{9}{23} \\ \frac{9}{23} & \frac{3}{23} & -\frac{2}{23} \end{bmatrix} \\
&= \frac{1}{\det D_4} \begin{bmatrix} P_2^2 - P_1 P_3 & P_0 P_3 - P_1 P_2 & P_1^2 - P_0 P_2 \\ P_1^2 - P_0 P_2 & P_{-1} P_2 - P_0 P_1 & P_0^2 - P_{-1} P_1 \\ P_0 P_3 - P_2 P_1 & P_1^2 - P_{-1} P_3 & P_{-1} P_2 - P_1 P_0 \end{bmatrix} \\
&= \frac{1}{23} \begin{bmatrix} 4 & 9 & 6 \\ -6 & -2 & 9 \\ 9 & 3 & -2 \end{bmatrix}.
\end{aligned}
$$

Let us represent the initial message in the form of the square matrix M of order 3. If the message is not a 3rd order matrix, fill the empty matrix elements with zeros, exclamation marks, question marks or dots. Then, substitute values from Table 2 for the characters within the alphabet table according to our preferences. This table is organized based on $mod30$ for a chosen arbitrary value of l. Note that we start by selecting the first character as "l".

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
l	$l+1$	$l+2$	$l+3$	$l+4$	$l+5$	$l+6$	$l+7$	$l+8$	$l+9$
K	**L**	**M**	**N**	**O**	**P**	**Q**	**R**	**S**	**T**
$l+10$	$l+11$	$l+12$	$l+13$	$l+14$	$l+15$	$l+16$	$l+17$	$l+18$	$l+19$
U	**V**	**W**	**X**	**Y**	**Z**	**0**	**!**	**?**	**.**
$l+20$	$l+21$	$l+22$	$l+23$	$l+24$	$l+25$	$l+26$	$l+27$	$l+28$	$l+29$

Table 18: *Alphabet Table*

This method encrypts the message matrix M, by multiplying D_n given by (2.3), i.e. $M \times D_n = C$ where C is the coded matrix. Conversely, the transformation $C \times (D_n)^{-1} = M$ gives us to decrypt the encoded matrix which is the messsage matrix.

3 Connections Between The Code Matrix Elements

At this section, we give the Perrin coding/decoding approach specifically designed for a 3×3 matrix. A noteworthy relationship exists among the elements within a code matrix C, playing an important role in the error-correction procedure. We can formulate the following equations to represent the procedures for coding and decoding. Concerning the encoding message:

$$C = M \times D_n = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \times \begin{bmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \tag{3.1}$$

$$c_1 = m_1P_{n-5} + m_2P_{n-4} + m_3P_{n-3}$$
$$c_2 = m_1P_{n-3} + m_2P_{n-2} + m_3P_{n-1}$$
$$c_3 = m_1P_{n-4} + m_2P_{n-3} + m_3P_{n-2}$$
$$c_4 = m_4P_{n-5} + m_5P_{n-4} + m_6P_{n-3}$$
$$c_5 = m_4P_{n-3} + m_5P_{n-2} + m_6P_{n-1}$$
$$c_6 = m_4P_{n-4} + m_5P_{n-3} + m_6P_{n-2}$$
$$c_7 = m_7P_{n-5} + m_8P_{n-4} + m_9P_{n-3}$$
$$c_8 = m_7P_{n-3} + m_8P_{n-2} + m_9P_{n-1}$$
$$c_9 = m_7P_{n-4} + m_8P_{n-3} + m_9P_{n-2}$$

Given that both the matrix and the message matrix comprise non-negative integers, it follows that all elements within the matrix are non-negative integers.

We can describe the matrix used in the decoding process as follows:

$$M = C \times (D_n)^{-1} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \times \begin{bmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}^{-1} \tag{3.2}$$

$$= \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \tag{3.3}$$

where the inverse of (D_n) given in Theorem 2.3. In the instance of a positive integer represented by n, the corresponding equation can be expressed as follows:

$$\begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \times (D_n)^{-1} \tag{3.4}$$

since the det $(Q^n) = 1$ and det $(K) = 23$. Derived from equation (7), it can be inferred that the elements constituting the matrix M can be acquired using the subsequent formulas. Considering the variable $l \geq 0$, the elements that constitute the matrix M are as follows:

$$m_1 \geq 0, m_2 \geq 0, m_3 \geq 0, m_4 \geq 0, m_5 \geq 0, m_6 \geq 0, m_7 \geq 0, m_8 \geq 0, m_9 \geq 0 \tag{3.5}$$

From equation (7), one can see that the following equations hold:

$$m_1 = \frac{1}{23}\left((P_{n-2}^2 - P_{n-3}P_{n-1})c_1 + (P_{n-3}^2 - P_{n-4}P_{n-2})c_2 + (P_{n-4}P_{n-1} - P_{n-2}P_{n-3})c_3\right) \tag{3.6}$$

$$m_2 = \frac{1}{23}\left((P_{n-4}P_{n-1} - P_{n-3}P_{n-2})c_1 + (P_{n-5}P_{n-2} - P_{n-4}P_{n-3})c_2 + (P_{n-3}^2 - P_{n-5}P_{n-1})c_3\right) \tag{3.7}$$

$$m_3 = \frac{1}{23}\left((P_{n-3}^2 - P_{n-4}P_{n-2})c_1 + (P_{n-4}^2 - P_{n-5}P_{n-3})c_2 + (P_{n-5}P_{n-2} - P_{n-3}P_{n-4})c_3\right) \tag{3.8}$$

$$m_4 = \frac{1}{23}\left((P_{n-2}^2 - P_{n-3}P_{n-1})c_4 + (P_{n-3}^2 - P_{n-4}P_{n-2})c_5 + (P_{n-4}P_{n-1} - P_{n-2}P_{n-3})c_6\right) \tag{3.9}$$

$$m_5 = \frac{1}{23}\left((P_{n-4}P_{n-1} - P_{n-3}P_{n-2})c_4 + (P_{n-5}P_{n-2} - P_{n-4}P_{n-3})c_5 + (P_{n-3}^2 - P_{n-5}P_{n-1})c_6\right) \tag{3.10}$$

$$m_6 = \frac{1}{23}\left((P_{n-3}^2 - P_{n-4}P_{n-2})c_4 + (P_{n-4}^2 - P_{n-5}P_{n-3})c_5 + (P_{n-5}P_{n-2} - P_{n-3}P_{n-4})c_6\right) \tag{3.11}$$

$$m_7 = \frac{1}{23}\left((P_{n-2}^2 - P_{n-3}P_{n-1})c_7 + (P_{n-3}^2 - P_{n-4}P_{n-2})c_8 + (P_{n-4}P_{n-1} - P_{n-2}P_{n-3})c_9\right) \tag{3.12}$$

$$m_8 = \frac{1}{23}\left((P_{n-4}P_{n-1} - P_{n-3}P_{n-2})c_7 + (P_{n-5}P_{n-2} - P_{n-4}P_{n-3})c_8 + (P_{n-3}^2 - P_{n-5}P_{n-1})c_9\right) \tag{3.13}$$

$$m_9 = \frac{1}{23}\left((P_{n-3}^2 - P_{n-4}P_{n-2})c_7 + (P_{n-4}^2 - P_{n-5}P_{n-3})c_8 + (P_{n-5}P_{n-2} - P_{n-3}P_{n-4})c_9\right) \tag{3.14}$$

By dividing both sides of the equations involving $m_1, m_2,$ and m_3 by a positive constant $c_1 > 0$, we obtain:

$$\frac{c_3}{c_1}\left(P_{n-4}P_{n-1} - P_{n-2}P_{n-3}\right) \geq \frac{c_2}{c_1}\left(P_{n-4}P_{n-2} - P_{n-2}^2\right) + \left(P_{n-3}P_{n-1} - P_{n-2}^2\right), \tag{3.15}$$

$$\frac{c_3}{c_1}\left(P_{n-5}P_{n-1} - P_{n-3}^2\right) \leq \frac{c_2}{c_1}\left(P_{n-5}P_{n-2} - P_{n-4}P_{n-3}\right) + \left(P_{n-4}P_{n-1} - P_{n-3}P_{n-2}\right) \tag{3.16}$$

and

$$\frac{c_3}{c_1}\left(P_{n-5}P_{n-2} - P_{n-3}P_{n-4}\right) \geq \quad \frac{c_2}{c_1}\left(P_{n-5}P_{n-3} - P_{n-4}^2\right) + \left(P_{n-4}P_{n-2} - P_{n-3}^2\right). \quad (3.17)$$

Let

$$A_1 = \left(P_{n-4}P_{n-1} - P_{n-2}P_{n-3}\right),$$
$$A_2 = \left(P_{n-5}P_{n-1} - P_{n-3}^2\right),$$
$$A_3 = \left(P_{n-5}P_{n-2} - P_{n-3}P_{n-4}\right).$$

Exploring the various possibilities arising from $3^3 = 27$ cases involving $A_1 \geq 0, A_2 \geq 0$, and $A_3 \geq 0$, we delve into discussions regarding a selection of these 27 cases. We examine three cases since the similarities of the other cases.

- **Case 1.** If $A_1 > 0, A_2 > 0, A_3 > 0$: By (18), we have:

$$\frac{c_3}{c_1} \geq u \quad (3.18)$$

where

$$u = \frac{c_2}{c_1}\left(\frac{P_{n-4}P_{n-2} - P_{n-2}^2}{A_1}\right) + \left(\frac{P_{n-3}P_{n-1} - P_{n-2}^2}{A_1}\right).$$

From (19), we get

$$\frac{c_3}{c_1} \leq v \quad (3.19)$$

where

$$v = \frac{c_2}{c_1}\left(\frac{P_{n-5}P_{n-2} - P_{n-4}P_{n-3}}{A_2}\right) + \left(\frac{P_{n-4}P_{n-1} - P_{n-3}P_{n-2}}{A_2}\right).$$

From (20) we have

$$\frac{c_3}{c_1} \geq w \quad (3.20)$$

where

$$w = \frac{c_2}{c_1}\left(\frac{P_{n-5}P_{n-3} - P_{n-4}^2}{A_3}\right) + \left(\frac{P_{n-4}P_{n-2} - P_{n-3}^2}{A_3}\right).$$

From (21) and (22), we have

$$\frac{c_1}{c_2} \geq \min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}. \quad (3.21)$$

Therefore, considering (21) and (23), we obtain:

$$\frac{c_1}{c_2} \leq \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}. \tag{3.22}$$

Thus,

$$\min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\} \leq \frac{c_1}{c_2} \quad \leq \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\} \tag{3.23}$$

$$\min\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\} \leq \frac{c_2}{c_3} \quad \leq \max\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\} \tag{3.24}$$

and

$$\min\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\} \leq \frac{c_1}{c_3} \quad \leq \max\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\}. \tag{3.25}$$

- **Case 2.** If $A_1 = 0$ and $A_2 > 0, A_3 > 0$: By (17), we have:

$$\frac{c_1}{c_2} \geq \min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}. \tag{3.26}$$

Since $A_1 = 0$, from (18) and (19), we obtain

$$\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}. \tag{3.27}$$

From (29) and (30), we have

$$\min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\} \leq \frac{c_1}{c_2} \quad \leq \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}.$$

- **Case 3.** If $A_1 < 0, A_2 < 0, A_3 < 0$: By (17), we have

$$\frac{c_3}{c_1} \leq u$$

where

$$u = \frac{c_2}{c_1}\left(\frac{P_{n-4}P_{n-2} - P_{n-2}^2}{A_1}\right) + \left(\frac{P_{n-3}P_{n-1} - P_{n-2}^2}{A_1}\right).$$

From (18), we get

$$\frac{c_3}{c_1} \geq v$$

where

$$v = \frac{c_2}{c_1}\left(\frac{P_{n-5}P_{n-2} - P_{n-4}P_{n-3}}{A_2}\right) + \left(\frac{P_{n-4}P_{n-1} - P_{n-3}P_{n-2}}{A_2}\right).$$

From (19), we have

$$\frac{c_3}{c_1} \le w$$

where

$$w = \frac{c_2}{c_1}\left(\frac{P_{n-5}P_{n-3} - P_{n-4}^2}{A_3}\right) + \left(\frac{P_{n-4}P_{n-2} - P_{n-3}^2}{A_3}\right).$$

From (26) and (27), we have

$$\frac{c_1}{c_2} \le \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}$$

Hence, from (26) and (28), we have

$$\frac{c_1}{c_2} \ge \min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}.$$

Thus,

$$\min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\} \le \frac{c_1}{c_2} \quad \le \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}.$$

Similarly, we have

$$\min\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\} \le \frac{c_2}{c_3} \quad \le \max\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\}$$

and

$$\min\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\} \le \frac{c_1}{c_3} \quad \le \max\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\}.$$

Similarly it can be proved for the rest cases. Hence, we have

$$\min\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\} \le \frac{c_1}{c_2} \quad \le \max\left\{\frac{P_{n-5}}{P_{n-3}}, \frac{P_{n-4}}{P_{n-2}}, \frac{P_{n-3}}{P_{n-1}}\right\}$$

$$\min\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\} \le \frac{c_2}{c_3} \quad \le \max\left\{\frac{P_{n-3}}{P_{n-4}}, \frac{P_{n-2}}{P_{n-3}}, \frac{P_{n-1}}{P_{n-2}}\right\}$$

and

$$\min\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\} \leq \frac{c_1}{c_3} \quad \leq \max\left\{\frac{P_{n-5}}{P_{n-4}}, \frac{P_{n-4}}{P_{n-3}}, \frac{P_{n-3}}{P_{n-2}}\right\}.$$

Therefore, for large value of n, we get

$$\frac{c_1}{c_2} \approx \frac{1}{\alpha^2}, \quad \frac{c_2}{c_3} \approx \alpha \quad \text{and} \quad \frac{c_1}{c_3} \approx \frac{1}{\alpha}$$

where $\alpha = 1.61803398$. Similarly, we have

$$\frac{c_4}{c_5} \approx \frac{1}{\alpha^2}, \quad \frac{c_5}{c_6} \approx \alpha \quad \text{and} \quad \frac{c_4}{c_6} \approx \frac{1}{\alpha}$$

and

$$\frac{c_7}{c_8} \approx \frac{1}{\alpha^2}, \quad \frac{c_8}{c_9} \approx \alpha \quad \text{and} \quad \frac{c_7}{c_9} \approx \frac{1}{\alpha}.$$

4 Illustrative Examples

Example 4.1. Let's consider the following text:

"ALGORITHM"

- Step 1: Let's form the message matrix:

$$M = \begin{bmatrix} A & L & G \\ O & R & I \\ T & H & M \end{bmatrix}_{3\times 3}$$

- Step 2: By taking into account the Table 2, and choosing arbitrary value " l ", here we consider $l = 4$, we get:

$$M = \begin{bmatrix} 4 & 15 & 10 \\ 18 & 21 & 12 \\ 23 & 11 & 16 \end{bmatrix}$$

- Step 3: For $n = 3$, we have;

$$\begin{aligned} D_3 &= KQ^3 \\ &= \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 3 \\ 3 & 2 & 0 \end{bmatrix}. \end{aligned}$$

- Step 4: The coded message is:

$$\begin{aligned} C &= M \times D_3 \\ &= \begin{bmatrix} 4 & 15 & 10 \\ 18 & 21 & 12 \\ 23 & 11 & 16 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 3 \\ 3 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 35 & 67 & 30 \\ 65 & 18 & 38 \\ 48 & 87 & 54 \end{bmatrix} = \begin{bmatrix} E & G & A \\ E & R & H \\ R & 0 & X \end{bmatrix}. \end{aligned}$$

- Step 5: To decode message, multiply it by the inverse matrix of D_3, in other words:

$$\begin{aligned} M &= C \times (D_3)^{-1} \\ &= \begin{bmatrix} 35 & 67 & 30 \\ 65 & 18 & 38 \\ 48 & 87 & 54 \end{bmatrix} \begin{bmatrix} -\frac{6}{23} & -\frac{2}{23} & \frac{9}{23} \\ \frac{9}{23} & \frac{3}{23} & -\frac{2}{23} \\ -\frac{2}{23} & \frac{7}{23} & \frac{3}{23} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 15 & 10 \\ 18 & 21 & 12 \\ 23 & 11 & 16 \end{bmatrix} = \begin{bmatrix} A & L & G \\ O & R & I \\ T & H & M \end{bmatrix}. \end{aligned}$$

Example 4.2. Let's consider the following text:

"QUADRATIC"

- Step 1: Let's form the message matrix:

$$M = \begin{bmatrix} Q & U & A \\ D & R & A \\ T & I & C \end{bmatrix}_{3\times 3}$$

- Step 2: By taking into account the Table 2, and choosing arbitrary value " l ", here we consider $l = 6$, we get :

$$M = \begin{bmatrix} 22 & 26 & 6 \\ 9 & 23 & 6 \\ 25 & 14 & 8 \end{bmatrix}.$$

- Step 3: For $n = 4$, we have:

$$\begin{aligned} D_4 &= KQ^4 \\ &= \begin{bmatrix} -1 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}. \end{aligned}$$

- Step 4: The coded message is:

$$\begin{aligned} C &= M \times D_4 \\ &= \begin{bmatrix} 22 & 26 & 6 \\ 9 & 23 & 6 \\ 25 & 14 & 8 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 53 & 16 & 18 \\ 84 & 124 & 30 \\ 77 & 97 & 34 \end{bmatrix} = \begin{bmatrix} W & P & R \\ X & D & A \\ Q & G & D \end{bmatrix}. \end{aligned}$$

- To decode message, multiply it by the inverse matrix of D_4, in other words:

$$\begin{aligned} M &= C \times (D_4)^{-1} \\ &= \begin{bmatrix} 53 & 16 & 18 \\ 84 & 124 & 30 \\ 77 & 97 & 34 \end{bmatrix} \begin{bmatrix} \frac{4}{23} & \frac{9}{23} & -\frac{6}{23} \\ -\frac{6}{23} & -\frac{2}{23} & \frac{9}{23} \\ \frac{9}{23} & \frac{3}{23} & -\frac{2}{23} \end{bmatrix} \\ &= \begin{bmatrix} 22 & 26 & 6 \\ 9 & 23 & 6 \\ 25 & 14 & 8 \end{bmatrix} = \begin{bmatrix} Q & U & A \\ D & R & A \\ T & I & C \end{bmatrix}. \end{aligned}$$

Example 4.3. Let's consider the following text:

"RING"

- Step 1: Let's form the message matrix:

$$M = \begin{bmatrix} R & I & N \\ G & 0 & ! \\ ? & 0 & ! \end{bmatrix}_{3\times 3}$$

- Step 2: By taking into account the Table , and choosing arbitrary value " l ", here we consider $l = 5$, we get :

$$M = \begin{bmatrix} 22 & 13 & 18 \\ 11 & 31 & 32 \\ 33 & 31 & 32 \end{bmatrix}.$$

- Step 3: For $n = 5$, we have:

$$D_5 = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

- Step 4: The coded message is:

$$\begin{aligned} C &= M \times D_5 \\ &= \begin{bmatrix} 22 & 13 & 18 \\ 11 & 31 & 32 \\ 33 & 31 & 32 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 102 & 119 & 80 \\ 97 & 179 & 158 \\ 163 & 223 & 158 \end{bmatrix} = \begin{bmatrix} L & ? & T \\ G & ? & H \\ M & M & H \end{bmatrix}. \end{aligned}$$

- To decode message, multiply it by the inverse matrix of D_4, in other words:

$$\begin{aligned} M &= C \times (D_5)^{-1} \\ &= \begin{bmatrix} 102 & 119 & 80 \\ 97 & 179 & 158 \\ 163 & 223 & 158 \end{bmatrix} \begin{bmatrix} \frac{5}{23} & -\frac{6}{23} & \frac{4}{23} \\ \frac{4}{23} & \frac{9}{23} & -\frac{6}{23} \\ -\frac{6}{23} & -\frac{2}{23} & \frac{9}{23} \end{bmatrix} \\ &= \begin{bmatrix} 22 & 13 & 18 \\ 11 & 31 & 32 \\ 33 & 31 & 32 \end{bmatrix} = \begin{bmatrix} R & I & N \\ G & 0 & ! \\ ? & 0 & ! \end{bmatrix}. \end{aligned}$$

5 Conclusion

The primary utilization of Perrin numbers and their matrix representation is the basic idea of this coding/decoding method. This method stands apart from classical algebraic codes due to distinct characteristics with its most crucial feature being the capacity for error correction. The Perrin coding/decoding method transforms into matrix multiplication and this method ensures the complete recovery of all flawed 3×3 code matrices, whether they contain single-fold, double-fold, all the way up to eight-fold errors.

References

[1] Adegoke, K. (2018). Summation identities involving Padovan and Perrin numbers. arXiv preprint arXiv:1812.03241.

[2] Bravo, E. F., Bravo, J. J., & Luca, F. (2022). On the multiplicities of Padovan-type sequences. Bulletin of the Brazilian Mathematical Society, New Series, 53(4), 1519-1549.

[3] Laisant, C. A., & Lemoine, E. M. H. (Eds.). (1894). L'Intermédiaire des mathematiciens (Vol. 1). Gauthier-Villars et Fils.

[4] Lucas, E. (1878). Théorie des fonctions numériques simplement périodiques. American Journal of Mathematics, 289-321.

[5] Shtayat, J., & Al-Kateeb, A. A. (2022). The Perrin R-matrix and more properties with an application. Journal of Discrete Mathematical Sciences and Cryptography, 25(1), 41-52.

[6] N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).

[7] Sokhuma, K. (2013). Padovan q-matrix and the generalized relations. Applied Mathematical Sciences, 7(56), 2777-2780.

[8] Yilmaz, N., & Taskara, N. (2013). Matrix sequences in terms of Padovan and Perrin numbers. Journal of Applied Mathematics, 2013(1), 941673.

[9] Shannon, A. G., Anderson, P. G., & Horadam, A. F. (2006). Properties of Cordonnier, Perrin and van der Laan numbers. International Journal of Mathematical Education in Science and Technology, 37(7), 825-831.